Lipschitz functions with maximal Clarke subdifferentials are staunch

Jonathan M. Borwein[1] and Xianfu Wang[2]

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ABSTRACT. In a recent paper we have shown that most non-expansive Lipschitz functions (in the sense of Baire's category) have a maximal Clarke subdifferential. In the present paper, we show that in a separable Banach space the set of non-expansive Lipschitz functions with a maximal Clarke subdifferential is not only of generic, but also staunch.

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1 Introduction and Definitions

Lipschitz functions with maximal subdifferentials provide counter-examples in nonsmooth analysis and differentiability theory. In a recent paper [1], we showed that the set of Lipschitz functions with maximal subdifferentials is residual in the space of all non-expansive functions. The purpose of this note is to strengthen this by showing that, in a separable-setting the set of all non-expansive Lipschitz functions with maximal subdifferentials is not only of residual but also *staunch*, by which we mean the complement of the set is σ-porous. We now recall the appropriate notion of porosity.

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called *porous* in (Y, d) if there exist $0 < \alpha \leq 1$ and $r_0 > 0$ such that for each $0 < r \leq r_0$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E. \tag{1}$$

A subset of the space Y is called *σ-porous* in (Y, d) if it is a countable union of porous subets in (Y, d). All σ-porous sets are of the first category. If Y is a finite dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. The class of σ-porous sets is much smaller than the class of sets which have measure 0 and are of the first category. In fact, every complete metric space without isolated points contains a closed nowhere dense set which is not σ-porous [6].

Throughout, X is a separable Banach space with norm $\|\cdot\|$, and its topological dual is denoted by X^* with dual unit ball B^*. We use S_X to denote the unit sphere of X. Let $A \subset X$ be a bounded open convex set. For a real-valued $f : A \to R$ we say that f is *K-Lipschitz* on A if $K > 0$ and $|f(x) - f(y)| \leq K\|x - y\|$ for all $x, y \in A$. When $K = 1$, f is called *nonexpansive*. The *Clarke derivative* of f at point x in the direction v is given by

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

while the *Clarke subdifferential* $\partial_c f$ is given by:

$$\partial_c f(x) := \{x^* \in X^* | \ \langle x^*, v\rangle \leq f^\circ(x; v) \text{ for all } v \in X\}.$$

Note that $f^\circ(x; v)$ is upper semicontinuous as a function of (x, v). Being nonempty and weak* compact convex valued, the multifunction $\partial_c f : A \to 2^{X^*}$ is norm-to-weak* upper semicontinuous. Detailed properties about Clarke subdifferentials can be found in [3], which is a sort of bible for nonsmooth analysts.

2 The Main Result

Let C be a weak*–compact convex subset of X^*. Recall that the *support function* of C is the function $\sigma_C : X \to R$ defined by

$$\sigma_C(v) := \sup\{\langle x^*, v\rangle | \ x^* \in C\}.$$

σ_C is sublinear, and Lipschitz with Lipschitz rate $K := \sup\{\|x^*\| : \ x^* \in C\}$. Consider

$$\mathcal{N}_C := \{f | \ f : A \to R \text{ and } f(x) - f(y) \leq \sigma_C(x - y) \text{ for all } x, y \in A\}.$$

Since each $f \in \mathcal{N}_C$ satisfies $f(x) - f(y) \leq K\|x - y\|$ for all $x, y \in A$, $\mathcal{N}_C$ is a special class of K–Lipschitz functions defined on A.

For $f, g \in \mathcal{N}_C$, set

$$\rho(f, g) := \sup_{x \in A} |f(x) - g(x)|.$$

One can easily verify that $(\mathcal{N}_C, \rho)$ is a complete metric space.

Our central result may now be stated.

Theorem 1 *Assume that X is a separable Banach space and let $A \subset X$ be a bounded open convex subset of X. In the complete metric space $(\mathcal{N}_C, \rho)$, there exists a subset G such that $\mathcal{N}_C \setminus G$ is σ-porous in $(\mathcal{N}_C, \rho)$, and such that each $f \in G$ has $\partial_c f \equiv C$ on A.*

Proof. Fix $x \in A$, $v \in S_X$ and a natural number k. Consider

$$G(x, v, k) := \left\{ f \in \mathcal{N}_C | \ \frac{f(x + tv) - f(x)}{t} - \sigma_C(v) \geq -\frac{1}{k} \text{ for some } 0 < t < \tfrac{1}{k} \right\}.$$

We shall show that $\mathcal{N}_C \setminus G(x, v, k)$ is porous in $(\mathcal{N}_C, \rho)$.

According to (1), it suffices to find $0 < \alpha \leq 1$ such that for each $r \in (0, 1/k)$ and each $f \in \mathcal{N}_C$ there exists $h_2 \in \mathcal{N}_C$ for which

$$B(h_2, \alpha r) \subset B(f, r) \cap G(x, v, k).$$

Of course, here h_2 relies on r, but α only relies on (x, v, k).

To meet this goal, we define $h : X \to R$ by

$$h(\tilde{x}) := f(x) - \frac{r}{4} + \sigma_C(\tilde{x} - x),$$

and set

$$h_1 := \min\{f, h\}, \quad h_2 := \max\{f - \frac{r}{2}, h_1\}. \tag{2}$$

Clearly, $h_2 \in \mathcal{N}_C$ and $f - r/2 \leq h_2 \leq f$, so that

$$\rho(h_2, f) \leq \frac{r}{2}.$$

Set

$$\alpha := \frac{\min\{d_{X \setminus A}(x), 1\}}{8(\sigma_C(v) + \sigma_C(-v) + 1)} \cdot \frac{1}{k}. \tag{3}$$

If we let

$$t := \frac{\min\{d_{X \setminus A}(x), 1\}}{4(\sigma_C(v) + \sigma_C(-v) + 1)} r, \tag{4}$$

where $d_{X \setminus A}(x) := \inf\{\|x - y\| : \ y \in X \setminus A\}$, then $0 < t < 1/k$ and $x + tv \in A$. Note that $d_{X \setminus A}(x) > 0$ because A is open and $x \in A$. Now

$$h(x + tv) = f(x) - \frac{r}{4} + t\sigma_C(v).$$

Since

$$f(x) - f(x + tv) \leq \sigma_C(-tv),$$

we have

$$f(x + tv) \geq f(x) - \sigma_C(-tv) = f(x) - t\sigma_C(-v).$$

The choice of t implies

$$t(\sigma_C(v) + \sigma_C(-v)) \leq \frac{r}{4},$$

so that

$$f(x) - \frac{r}{4} + t\sigma_C(v) \leq f(x) - t\sigma_C(-v).$$

It follows that $h(x + tv) \leq f(x + tv)$, and so $h_1(x + tv) = h(x + tv)$ by (2). On the other hand,

$$f(x + tv) - \frac{r}{2} \leq f(x) - \frac{r}{4} + t\sigma_C(v),$$

since $f(x + tv) - f(x) \leq \sigma_C(tv)$. Therefore, by (2),

$$h_2(x + tv) = f(x) - \frac{r}{4} + t\sigma_C(v) \quad \text{and } h_2(x) = f(x) - \frac{r}{4}.$$

This means

$$\frac{h_2(x + tv) - h_2(x)}{t} = \sigma_C(v). \tag{5}$$

Assume that $g \in B(h_2, \alpha r)$. We will show that $g \in G(x, v, k)$. Indeed, by (5), (4), (3),

$$\begin{aligned}
&\frac{g(x + tv) - g(x)}{t} - \sigma_C(v) \\
&= \frac{(g - h_2)(x + tv) - (g - h_2)(x)}{t} + \frac{h_2(x + tv) - h_2(x)}{t} - \sigma_C(v) \\
&\geq \frac{-2\alpha r}{t} = -2\alpha r t^{-1} = -2\alpha r \left[\frac{\min\{d_{X\setminus A}(x), 1\}}{4(\sigma_C(v) + \sigma_C(-v) + 1)} r\right]^{-1} \\
&= -\alpha \cdot \frac{8(\sigma_C(v) + \sigma_C(-v) + 1)}{\min\{d_{X\setminus A}(x), 1\}} = -\frac{1}{k}.
\end{aligned}$$

Therefore,

$$\{g \in \mathcal{N}_C : \rho(g, h_2) \leq \alpha r\} \subset G(x, v, k). \tag{6}$$

If $\rho(g, h_2) \leq \alpha r$, then

$$\rho(g, f) \leq \rho(g, h_2) + \rho(h_2, f) \leq \alpha r + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} = r.$$

Thus

$$\{g \in \mathcal{N}_C : \ \rho(g, h_2) \leq \alpha r\} \subset \{g \in \mathcal{N}_C : \ \rho(g, f) \leq r\}.$$

When combined with (6), this inclusion implies that

$$\mathcal{N}_C \setminus G(x, v, k) \quad \text{is indeed porous in } (\mathcal{N}_C, \rho). \tag{7}$$

Now let $\{x_n : n \geq 1\}$ be norm dense in A, $\{v_m : m \geq 1\}$ be norm dense in S_X. Set

$$G := \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} G(x_n, v_m, k).$$

In view of (7) and that

$$\mathcal{N}_C \setminus G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} (\mathcal{N}_C \setminus G(x_n, v_m, k)),$$

the set $\mathcal{N}_C \setminus G$ must be σ-porous in $(\mathcal{N}_C, \rho)$. If $f \in G$, then for each x_n, v_m, k, we have $f \in G(x_n, v_m, k)$; that is,

$$\frac{f(x_n + t_{n,m,k} v_m) - f(x_n)}{t_{n,m,k}} - \sigma_C(v_m) \geq -\frac{1}{k},$$

for some $0 < t_{n,m,k} < \frac{1}{k}$. When $k \to \infty$, from the definition of f° it follows that

$$f^\circ(x_n; v_m) \geq \limsup_{t \downarrow 0} \frac{f(x_n + t v_m) - f(x_n)}{t} \geq \sigma_C(v_m).$$

For every $x \in A$ and $v \in S_X$, we may find subsequences (x_n) and (v_m) such that $x_n \to x$ and $v_m \to v$. By the upper semicontinuity of f° and continuity of σ_C, we get

$$f^\circ(x; v) \geq \sigma_C(v). \tag{8}$$

Since $f \in \mathcal{N}_C$, for every $y \in A, t > 0$,

$$f(y + tv) - f(y) \leq \sigma_C(tv).$$

Dividing both sides by t, and taking the lim sup as $y \to x$ and $t \downarrow 0$ produces

$$f^\circ(x; v) \leq \sigma_C(v).$$

Together with (8), we obtain

$$f^\circ(x; v) = \sigma_C(v) \quad \text{for } x \in A, v \in S_X.$$

Dually, $\partial_c f(x) = C$ for every $x \in A$, and the proof of the theorem is complete. □

Observe that

$$\mathcal{N}_{B^*} := \{f |\ f : A \to R \text{ is nonexpansive with respect to } \|\cdot\|\}.$$

Theorem 1 gives:

Corollary 1 *In the space of nonexpansive functions, $(\mathcal{N}_{B^*}, \rho)$, the set*

$$\{f \in \mathcal{N}_{B^*} |\ \partial_c f \equiv B^* \text{ on } A\},$$

has a σ-porous complement in $(\mathcal{N}_{B^*}, \rho)$.

It is well-known that every locally Lipschitz function f on an open subset A of a separable Banach space X is Gâteaux differentiable everywhere on A except for possibly a Haar-null subset. We need a result due to Giles and Sciffer [4].

Lemma 1 *Let $f : A \to R$ be a locally Lipschitz function on an open subset A of a separable Banach space X. Then the set*

$$\{x \in A |\ f^+(x; v) = f^\circ(x; v) \quad \textit{for all } v \in X\},$$

is residual in A. Here

$$f^+(x; v) := \limsup_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Combining Corollary 1 with Lemma 1 gives the following result.

Corollary 2 *In the space of nonexpansive functions, $(\mathcal{N}_{B^*}, \rho)$, the set*

$$\{f \in \mathcal{N}_{B^*} |\ f \textit{ is Gâteaux differentiable at most on a first category subset of } A\},$$

has a σ-porous complement in $(\mathcal{N}_{B^*}, \rho)$.

Proof. Let $f \in \mathcal{N}_{B^*}$ such that $\partial_c f \equiv B^*$ on A. Consider the set

$$S_f := \{x \in A |\ f^+(x; v) = f^\circ(x; v) \quad \text{for all } v \in X\}.$$

By Lemma 1, S_f is a residual set in A. If f is Gâteaux differentiable at x, then $f^+(x; v) = \langle \nabla f(x), v \rangle$ for every $v \in X$, and so $x \notin S_f$ since $\partial_c f(x) = B^*$. Therefore, such an f is at most Gâteaux differentiable on $A \setminus S_f$, which is a first category subset in A. Since the set

$$\{f \in \mathcal{N}_{B^*} |\ \partial_c f \equiv B^* \text{ on } A\},$$

has a σ-porous complement in $(\mathcal{N}_{B^*}, \rho)$ by Corollary 1, the result is proved. □

Finally, for various generic aspects of Lipschitz functions with maximal Clarke subdifferentials on general Banach spaces, we refer readers to [2]

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Faculty of Computer Science, Dalhousie University, 6050 University Avenue, Halifax, NS., Canada, B3H 1W5
Email: `jborwein@cs.dal.ca`

Department of Mathematics and Statistics, UBC Okanagan, 3333 University Way, Kelowna, BC., Canada V1V 1V7
E-mail: `Shawn.Wang@ubc.ca`