THE EVALUATION OF BESSEL FUNCTIONS VIA EXP-ARC INTEGRALS

DAVID BORWEIN, JONATHAN M. BORWEIN[1], and O-YEAT CHAN[2]

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Abstract. A standard method for computing values of Bessel functions has been to use the well-known ascending series for small argument, and to use an asymptotic series for large argument; with the choice of the series changing at some appropriate argument magnitude, depending on the number of digits required. In a recent paper, D. Borwein, J. Borwein, and R. Crandall [1] derived a series for an "exp-arc" integral which gave rise to an absolutely convergent series for the J and I Bessel functions with integral order. Such series can be rapidly evaluated via recursion and elementary operations, and provide a viable alternative to the conventional ascending-asymptotic switching. In the present work, we extend the method to deal with Bessel functions of general (non-integral) order, as well as to deal with the Y and K Bessel functions.

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1. Introduction

Bessel functions are amongst the most important and most commonly occurring objects in mathematical physics. They arise as solutions to *Bessel's equation* [3, Eq. 10.2.1], [14, p. 38]

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+(z^2-\nu^2)y=0, \tag{1.1}$$

which is a special case of Laplace's equation under cylindrical symmetry. The *ordinary Bessel function of the first kind of order* ν is the solution $J_\nu(z)$ given by the ascending series [3, Eq. 10.2.2], [14, p. 40]

$$J_\nu(z):=\left(\frac{z}{2}\right)^\nu\sum_{m=0}^\infty\frac{(-1)^m(z/2)^{2m}}{m!\Gamma(\nu+m+1)}. \tag{1.2}$$

It is clear that although (1.2) converges rapidly for small $|z|$, it is computationally ineffective when $|z/2|^2$ is much greater than ν. One approach to overcoming this difficulty is to use the ascending series (1.2) for small $|z|$, and to use the asymptotic series below [3, Eq. 10.17.3], [14, p. 199] for large $|z|$:

$$J_\nu(z)\sim\left(\frac{2}{\pi z}\right)^{1/2}\left(\cos\omega\sum_{k=0}^\infty(-1)^k\frac{p_{2k}(\nu)}{(2z)^{2k}}-\sin\omega\sum_{k=0}^\infty(-1)^k\frac{p_{2k+1}(\nu)}{(2z)^{2k+1}}\right), \tag{1.3}$$

where

$$\omega:=z-\frac{\pi\nu}{2}-\frac{\pi}{4},$$

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and

$$p_k(\nu) := \frac{1}{k!}\prod_{m=1}^{k}\frac{4\nu^2-(2m-1)^2}{4} = \frac{(\frac{1}{2}-\nu)_k(\frac{1}{2}+\nu)_k}{k!},$$

with the empty product that arises at $k=0$ understood to be equal to 1 and

$$(a)_n := a(a+1)\cdots(a+n-1)$$

is the *Pochhammer* symbol. This approach is used in some texts on computation, for example in S. Zhang and J. Jin [15, p. 161]. However, one of the major drawbacks of using the asymptotic series (1.3) is that while it is known [14, p.206] that when $2N > \nu - \frac{1}{2}$, the error from truncating the right-hand side of (1.3) at the N-th term is bounded by the absolute value of the $N+1$-st term, the right-hand side of (1.3) is divergent for fixed z. Therefore, the use of (1.3) imposes upon us a theoretical limit on the number of correct digits that can be obtained, which in turn forces us to switch back to the ascending series (1.2) for very-high-precision computations.

The theory is similarly limited for the second (linearly independent of J_ν) solution of (1.1), known as the *ordinary Bessel function of the second kind* $Y_\nu(z)$, defined by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \tag{1.4}$$

for non-integral ν, and defined as the limit of the above expression at integral ν. In particular, we have the following expression for $Y_n(z)$, where n is an integer [14, pp. 62, 64].

$$\begin{aligned} Y_n(z) = \frac{1}{\pi}\Bigg(& 2(\log(z/2)+\gamma)J_n(z) - \sum_{k=0}^{n-1}\frac{(n-k-1)!(z/2)^{2k-n}}{k!} \\ & - \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k+n}(H_k+H_{k+n})}{k!(n+k)!}\Bigg), \end{aligned} \tag{1.5}$$

where

$$H_k := \sum_{j=1}^{k}\frac{1}{j}$$

is the k-th harmonic number. It also has an asymptotic expansion similar to (1.3).

$$Y_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2}\left(\sin\omega\sum_{k=0}^{\infty}(-1)^k\frac{p_{2k}(\nu)}{(2z)^{2k}} + \cos\omega\sum_{k=0}^{\infty}(-1)^k\frac{p_{2k+1}(\nu)}{(2z)^{2k+1}}\right), \tag{1.6}$$

with the same error bounds as indicated above.

Similar expansions exist for the *modified Bessel functions* $I_\nu(z)$ and $K_\nu(z)$. For completeness, we state their definitions below.

$$I_\nu(z) := e^{-\nu\pi i/2}J_\nu(iz) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty}\frac{(z/2)^{2m}}{m!\Gamma(\nu+m+1)}, \tag{1.7}$$

and

$$K_\nu(z) := \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}. \tag{1.8}$$

It should be noted, however, that there does exist a convergent asymptotic expansion for $I_\nu(z)$, due to Hadamard [14, p. 204].

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}\Gamma(\nu+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-\nu)_n}{n!(2z)^n}\gamma(\nu+n+\tfrac{1}{2},2z), \tag{1.9}$$

where

$$\gamma(a,z) := z^a\int_0^1 e^{-zs}s^{a-1}ds$$

is the *incomplete gamma* function. One should observe that for large n, the summands in (1.9) are of order $O(n^{-\nu-3/2})$ and so although the series converges absolutely, it only does so algebraically (i.e., at polynomial rate) in the number of summands.

In his extensive research into the theory of Hadamard expansions, R. B. Paris ([4]–[11]) found several variants of (1.9) that converge much more rapidly. In particular, in [9], he describes a general procedure for obtaining and accelerating Hadamard expansions that leads to the following geometrically convergent (i.e., at geometric rate) version of (1.9) [12].

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{p_k(\nu)}{(2z)^k}P(k+\nu+\tfrac{1}{2},z)+\frac{e^{-z\pm\pi i(\nu+\frac{1}{2})}}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{(-1)^k p_k(\nu)}{(2z)^k}P(k+\nu+\tfrac{1}{2},-z), \tag{1.10}$$

where

$$P(a,z) := \frac{\gamma(a,z)}{\Gamma(a)},$$

and the signs in the exponential are chosen depending on the sign of $\arg(z)$.

In this paper, we approach series expansions of Bessel functions from a different angle: through the evaluation of "exp-arc" integrals. The use of exp-arc integrals was motivated by the recent work of D. Borwein, J. Borwein, and R. Crandall [1] in which these integrals were used to obtain explicit error bounds for the asymptotic expansions of Laguerre polynomials. As a corollary of their results, they developed geometrically convergent series for the J and I Bessel functions at integral order, whose summands can be computed recursively using elementary operations. These series are redolent of the modified Hadamard series of Paris, but do not follow as special cases. Their independence from the Paris modifications is both theoretically and computationally interesting, and can be viewed as complementary to Paris' investigations. We generalise these ideas to obtain exp-arc series for Bessel functions of non-integral order and for the Bessel functions of the second kind.

At this point, perhaps a brief explanation of the term "exp-arc" is in order. Although originally (in [1]) exp-arc stood for "exponential-arcsine", in the present work we shall use the term to indicate any of the functions

$$e^{\operatorname{arcsin} z}, e^{\operatorname{arcsinh} z}, e^{\operatorname{arccos} z}, e^{\operatorname{arccosh} z}.$$

Thus, an *exp-arc integral* is an integral involving a power of any of the above exp-arc functions. The main idea here is to exploit the Taylor expansion of exp-arc functions to reduce exp-arc integrals to sums whose summands can be computed recursively, as the Taylor coefficients of exp-arc functions satisfy second-order linear recurrences.

The rest of the paper is outlined as follows. In Section 2, we use exp-arc integrals to prove our series for $J_\nu(z)$ in detail. In Section 3, we prove analogous formulas for the other three Bessel functions mentioned above. Then in Section 4, we give an analysis of the effectiveness of our series and derive explicit error bounds on the tails. Finally, in Section 5, we provide some numerical calculations and compare our series with the traditional computation schemes.

2. The Evaluation of $J_\nu(z)$

To obtain our series for the Bessel function J, we evaluate the following integral representation of $J_\nu(z)$ [14, p. 176], valid for $\mathrm{Re}(z) > 0$:

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu t - z\sin t)dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t - z\sinh t}dt. \tag{2.1}$$

The first integral has been dealt with in [1, Sec. 5]. We state the key result below.

Theorem 2.1 (Borwein-Borwein-Crandall)**.** *For any complex pair (p, q) and real numbers α, $\beta \in (-\pi, \pi)$, let*

$$\mathcal{I}(p, q, \alpha, \beta) := \int_\alpha^\beta e^{-iq\omega}e^{p\cos\omega}d\omega, \tag{2.2}$$

and

$$r_{2m+1}(\nu) := \nu\prod_{j=1}^m\left(\nu^2 + (2j-1)^2\right), \qquad r_{2m}(\nu) := \prod_{j=1}^m\left(\nu^2 + (2j-2)^2\right). \tag{2.3}$$

Then we have the absolutely convergent representation

$$\mathcal{I}(p, q, \alpha, \beta) = \frac{ie^p}{q}\sum_{k=0}^\infty \frac{r_{k+1}(-2iq)}{k!}\int_{\sin\frac{\alpha}{2}}^{\sin\frac{\beta}{2}} x^k e^{-2px^2}dx. \tag{2.4}$$

In particular, for the case where $(\alpha, \beta) = (-\pi/2, \pi/2)$, we have

$$\mathcal{I}(p, q) := \mathcal{I}(p, q, -\pi/2, \pi/2) = \frac{2ie^p}{q}\sum_{k=0}^\infty \frac{r_{2k+1}(-2iq)}{(2k)!}B_k(p), \tag{2.5}$$

with

$$\begin{aligned} B_k(p) := \int_0^{1/\sqrt{2}} x^{2k}e^{-2px^2}dx &= \frac{1}{2^{k+1}\sqrt{2}}\int_0^1 e^{-pu}u^{k-\frac{1}{2}}du \\ &= -\frac{e^{-p}}{p2^{k+1}\sqrt{2}} + \left(k - \frac{1}{2}\right)\frac{B_{k-1}(p)}{2}. \end{aligned} \tag{2.6}$$

From Theorem 2.1 we easily deduce that

$$\begin{aligned} \frac{1}{\pi}\int_0^\pi \cos(\nu t - z\sin t)dt &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{i(\nu t - z\cos t)}e^{i\nu\pi/2} + e^{-i(\nu t - z\cos t)}e^{-i\nu\pi/2}dt \\ &= \frac{1}{2\pi}\left(e^{-i\nu\pi/2}\mathcal{I}(iz, \nu) + e^{i\nu\pi/2}\mathcal{I}(-iz, -\nu)\right) \\ &= \frac{1}{2\pi}\left(e^{-i\nu\pi/2}\mathcal{I}(iz, \nu) + e^{i\nu\pi/2}\mathcal{I}(-iz, \nu)\right). \end{aligned} \tag{2.7}$$

Our goal, therefore, is to find a rapidly converging series for the second (infinite domain) integral in (2.1). If we let $s = \sinh t$ so that $dt = ds/\sqrt{1+s^2}$, we find that, for $\mathrm{Re}(z) > 0$,

$$\begin{aligned} \int_0^\infty e^{-\nu t - z\sinh t}dt &= \int_0^\infty \frac{e^{-zs}e^{-\nu\,\mathrm{arcsinh}\, s}}{\sqrt{1+s^2}}ds \\ &= -\frac{1}{\nu}e^{-zs}e^{-\nu\,\mathrm{arcsinh}\, s}\Big|_0^\infty - \frac{z}{\nu}\int_0^\infty e^{-zs}e^{-\nu\,\mathrm{arcsinh}\, s}ds \\ &= \frac{1}{\nu} - \frac{z}{\nu}\int_0^\infty e^{-zs}e^{-\nu\,\mathrm{arcsinh}\, s}ds \end{aligned} \tag{2.8}$$

and we are led to consider the integral

$$F(z,\nu) := \int_0^\infty e^{-zs} e^{-\nu \operatorname{arcsinh} s} ds.$$

To compute $F(z,\nu)$, we fix a positive integer N, subdivide $[0, N+\frac{1}{2}]$ into short intervals, and deal with the integral on each interval separately. To that end, for an integer $k \geq 0$ define

$$F_k(z,\nu) := \begin{cases} \displaystyle\int_0^{1/2} e^{-zs} e^{-\nu \operatorname{arcsinh} s} ds, & \text{if } k = 0, \\ \\ \displaystyle\int_{k-1/2}^{k+1/2} e^{-zs} e^{-\nu \operatorname{arcsinh} s} ds, & \text{if } k > 0, \end{cases} \tag{2.9}$$

and let

$$F_\infty(z,\nu) := \int_{N+1/2}^\infty e^{-zs} e^{-\nu \operatorname{arcsinh} s} ds. \tag{2.10}$$

For each $k > 0$, we shift the integral to $[-1/2, 1/2]$ and expand the exp-arc factor as a power series about zero, then integrate term-by-term. In an analogous way, for F_∞ we expand the exp-arc factor as a series at infinity and integrate term-by-term. Here, and throughout the rest of this article, such interchanges are justified by Abel's Limit Theorem [13, p. 425]. Since we are mainly interested in the computational aspect of the series, rather than explicit expressions we aim for recurrence relations among the summands. Thus, we make use of the following two lemmas.

Lemma 2.2. *For each integer $k \geq 0$ and any $\nu \in \mathbb{C}$, we may expand $e^{-\nu \operatorname{arcsinh}(k+s)}$ as a power series about $s = 0$ with radius of convergence $r = |i - k| = \sqrt{k^2+1}$. Moreover, the coefficients $a_n(k,\nu)$ given by*

$$e^{-\nu \operatorname{arcsinh}(k+s)} = \sum_{n=0}^\infty a_n(k,\nu) s^n$$

satisfy the recurrence relation

$$a_{n+2}(k,\nu) = \frac{1}{k^2+1} \left(\frac{(\nu^2 - n^2) a_n(k,\nu) - k(n+1)(2n+1) a_{n+1}(k,\nu)}{(n+1)(n+2)} \right), \tag{2.11}$$

with initial conditions

$$a_0(k,\nu) = (k + \sqrt{k^2+1})^{-\nu}, \qquad a_1(k,\nu) = -\frac{\nu a_0(k,\nu)}{\sqrt{k^2+1}}. \tag{2.12}$$

Proof. Since $e^{-\nu \operatorname{arcsinh}(k+s)}$ is analytic everywhere except when $k + s = \pm iy$, $y \in \mathbb{R}_{\geq 1}$, the Taylor expansion exists with radius of convergence as stated above. To compute the $a_n(k,\nu)$, let

$$f_k(s) := e^{-\nu \operatorname{arcsinh}(k+s)}.$$

Then one easily verifies that $f_k(s)$ satisfies the differential equation

$$f_k''(s) = \frac{1}{k^2 + 1 + 2ks + s^2} \left(\nu^2 f_k(s) - (k+s) f_k'(s) \right). \tag{2.13}$$

Clearing denominators and equating coefficients of s^n, we easily find that

$$n(n-1)a_n + 2k(n+1)na_{n+1} + (k^2+1)(n+2)(n+1)a_{n+2} = \nu^2 a_n - k(n+1)a_{n+1} - na_n.$$

Rearranging, we obtain

$$(k^2+1)(n+2)(n+1)a_{n+2} = (\nu^2-n^2)a_n - k(n+1)(2n+1)a_{n+1},$$

with $a_0 = f_k(0) = (k+\sqrt{k^2+1})^{-\nu}$ and $a_1 = f_k'(0) = \dfrac{-\nu}{\sqrt{k^2+1}}(k+\sqrt{k^2+1})^{-\nu}$. This is equivalent to (2.11). □

Lemma 2.3. *Recall that $(a)_n := a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. For $\nu \in \mathbb{C}$, the function $s^\nu e^{-\nu \operatorname{arcsinh} s}$ has the expansion*

$$s^\nu e^{-\nu \operatorname{arcsinh} s} = \sum_{n=0}^{\infty} \frac{A_n(\nu)}{s^{2n}}, \tag{2.14}$$

where $A_0(\nu) = 2^{-\nu}$ and for $n \geq 1$,

$$A_n(\nu) = \frac{(-1)^n \nu 2^{-\nu}(\nu+n+1)_{n-1}}{2^{2n} n!}, \tag{2.15}$$

provided that ν is not a negative integer. If ν is a negative integer, say $\nu = -m$, $m \in \mathbb{N}$, then (2.15) is valid for $1 \leq n < m$, and $A_n(-m) = (-1)^{m+1} A_{n-m}(m)$ for $n \geq m$. Note that this expansion is valid for $|s| > 1$.

Proof. Let

$$\begin{aligned} g(s) := s^\nu e^{-\nu \operatorname{arcsinh} s} &= s^\nu \left(s + \sqrt{1+s^2}\right)^{-\nu} \\ &= \left(1 + \sqrt{1+s^{-2}}\right)^{-\nu} \\ &= \left(1 + \sum_{k\geq 0} \binom{1/2}{k} s^{-2k}\right)^{-\nu}. \end{aligned}$$

When $|s| > 1$, we have $|1+s^{-2}| < 2$; so that $\left|\sum_{k\geq 1}\binom{1/2}{k} s^{-2k}\right| < 1$. Therefore

$$\begin{aligned} g(s) &= 2^{-\nu}\left(1 + \sum_{k\geq 1}\binom{1/2}{k}\frac{s^{-2k}}{2}\right)^{-\nu} \\ &= 2^{-\nu}\sum_{m\geq 0}\binom{-\nu}{m}\left(\sum_{k\geq 1}\binom{1/2}{k}\frac{1}{2s^{2k}}\right)^m \\ &= \sum_{n\geq 0}\frac{A_n(\nu)}{s^{2n}}, \end{aligned} \tag{2.16}$$

for some constants $A_n(\nu)$ with $A_0(\nu) = 2^{-\nu}$. Let us find a recurrence for $A_n(\nu)$. Applying (2.13) with $k = 0$ we find that

$$(1+s^2)\frac{d^2}{ds^2}(s^{-\nu}g(s)) = \nu^2 s^{-\nu} g(s) - s\frac{d}{ds}(s^{-\nu}g(s)).$$

Thus,

$$(2n-2+\nu)(2n-1+\nu)A_{n-1} + (2n+\nu)(2n+1+\nu)A_n = \nu^2 A_n - (2n+\nu)A_n.$$

Rearranging, we find that

$$((2n+\nu)^2 - \nu^2)A_n = -(\nu+2n-2)(\nu+2n-1)A_{n-1}, \tag{2.17}$$

and so, when ν is not a negative integer, we have

$$A_n = -\frac{(\nu+2n-2)(\nu+2n-1)}{4n(n+\nu)} A_{n-1}. \tag{2.18}$$

It is easy to verify that (2.15) solves this recurrence. If $\nu = -m$, $m \in \mathbb{N}$, then the left-hand side of (2.17) is zero when $n = m = -\nu$; thus, we need to find another expression for $A_n(-m)$ when $n \geq m$. However, note that

$$\begin{aligned}
\sum_{n=-m}^{\infty} \frac{A_{n+m}(-m)}{s^{2n}} + (-1)^m \sum_{n=0}^{\infty} \frac{A_n(m)}{s^{2n}} &= s^{2m} s^{-m} e^{m \operatorname{arcsinh} s} + (-1)^m s^m e^{-m \operatorname{arcsinh} s} \\
&= s^{2m} \left(1 + \sqrt{1+s^{-2}}\right)^m + (-1)^m \left(1 + \sqrt{1+s^{-2}}\right)^{-m} \\
&= s^{2m} \left(1 + \sqrt{1+s^{-2}}\right)^m + (-1)^m \left(s^2\left(-1 + \sqrt{1+s^{-2}}\right)\right)^m \\
&= s^m \left(\left(s + \sqrt{s^2+1}\right)^m + (-1)^m \left(-s + \sqrt{s^2+1}\right)^m\right) \\
&= s^m \left(\sum_{k=0}^{m} \binom{m}{k} s^k \left(\sqrt{s^2+1}\right)^{m-k} \left(1 + (-1)^{m-k}\right)\right)
\end{aligned}$$

is a polynomial in s where the smallest power of s is at least m. Therefore, we conclude that all the terms in powers of s^{-2} (including the constant term) are zero, and thus $A_{n+m}(-m) = -(-1)^m A_n(m)$ for $n \geq 0$, or equivalently, $A_n(-m) = (-1)^{m+1} A_{n-m}(m)$ for $n \geq m$. $\square$

We are now ready to write down our expression for $J_\nu(z)$.

Theorem 2.4 (**Exp-arc series for J_ν**)**.** *Let $z, \nu \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$, and let $N \in \mathbb{N}$. Then we have, when $\nu \neq 0$,*

$$\begin{aligned}
J_\nu(z) = \frac{1}{2\pi} &\left(e^{-i\nu\pi/2} \mathcal{I}(iz,\nu) + e^{i\nu\pi/2} \mathcal{I}(-iz,\nu)\right) \\
&+ \frac{z\sin(\nu\pi)}{\nu\pi} \left(-\frac{1}{z} + \sum_{n=0}^{\infty} \left(\alpha_n(z) a_n(0,\nu) + \beta_n(z) \sum_{k=1}^{N} e^{-kz} a_n(k,\nu)\right)\right. \\
&\left. + \sum_{n=0}^{\infty} A_n(\nu) I_n(N+\tfrac{1}{2}, z, \nu)\right),
\end{aligned} \tag{2.19}$$

where $\mathcal{I}(p,q)$ is given by (2.5), $a_n(k,\nu)$ and $A_n(\nu)$ are given by Lemmas 2.2 and 2.3, while

$$\alpha_n(z) := \int_0^{1/2} e^{-zs} s^n ds = -\frac{e^{-z/2}}{2^n z} + \frac{n}{z}\alpha_{n-1}(z), \tag{2.20}$$

$$\beta_n(z) := \int_{-1/2}^{1/2} e^{-zs} s^n ds = \frac{(-1)^n e^{z/2} - e^{-z/2}}{2^n z} + \frac{n}{z}\beta_{n-1}(z), \tag{2.21}$$

and

$$\begin{aligned}
I_n(\Theta, z, \nu) &:= \frac{e^{-\Theta z}}{\Theta^{2n+\nu-1}} \int_0^\infty e^{-\Theta z s}(1+s)^{-2n-\nu} ds \\
&= \frac{1}{(\nu+2n-1)(\nu+2n-2)} \left(\frac{e^{-\Theta z}(\nu+2n-2-\Theta z)}{\Theta^{2n+\nu-1}} + z^2 I_{n-1}(\Theta,z,\nu)\right).
\end{aligned} \tag{2.22}$$

For the case $\nu = 0$, we have

$$J_\nu(z) = \frac{1}{2\pi}\left(\mathcal{I}(iz,0) + \mathcal{I}(-iz,0)\right). \tag{2.23}$$

Proof. By (2.1), (2.7), and (2.8), it suffices to show that

$$\sum_{k=0}^{N} F_k(z,\nu) + F_\infty(z,\nu) = \sum_{n=0}^{\infty}\left(\alpha_n(z)a_n(0,\nu) + \beta_n(z)\sum_{k=1}^{N} e^{-kz}a_n(k,\nu)\right) + \sum_{n=0}^{\infty} A_n(\nu)I_n(N+\tfrac{1}{2},z,\nu).$$

For each k, we make a change of variable $s \mapsto k+s$ and expand $e^{-\nu\,\mathrm{arcsinh}(k+s)}$ as in Lemma 2.2. This yields

$$F_0(z,\nu) = \int_0^{1/2} e^{-zs}\sum_{n=0}^{\infty} a_n(0,\nu)s^n ds = \sum_{n=0}^{\infty}\alpha_n(z)a_n(0,\nu),$$

and, for $k \geq 1$,

$$F_k(z,\nu) = e^{-kz}\int_{-1/2}^{1/2} e^{-zs}\sum_{n=0}^{\infty} a_n(k,\nu)s^n ds = e^{-kz}\sum_{n=0}^{\infty} a_n(k,\nu)\beta_n(z).$$

For F_∞, we first expand $e^{-\nu\,\mathrm{arcsinh}\, s}$ as in Lemma 2.3 and then make a change of variable $s \mapsto (N+\frac{1}{2})(1+s)$. Thus

$$\begin{aligned} F_\infty(z,\nu) &= e^{-(N+1/2)z}\int_0^\infty e^{-(N+1/2)zs}\sum_{n=0}^{\infty}\frac{A_n(\nu)}{(N+\frac{1}{2})^{2n+\nu}(1+s)^{2n+\nu}}(N+\tfrac{1}{2})ds \\ &= \sum_{n=0}^{\infty} A_n(\nu)I_n(N+\tfrac{1}{2},z,\nu). \end{aligned}$$

The recurrence relations in (2.21) and (2.22) are easily obtained via integration by parts. $\square$

3. The Y, I, and K Bessel Functions

Using our results from Section 2 we obtain similar evaluations for the Bessel function of the second kind $Y_\nu(z)$, as well as for the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$. We make use of the integral representations [14, pp. 178, 181]:

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin t - \nu t)\ dt - \frac{1}{\pi}\int_0^\infty \left(e^{\nu t} + e^{-\nu t}\cos(\nu\pi)\right)e^{-z\sinh t}\ dt, \tag{3.1}$$

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos t}\cos(\nu t)\ dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}\ dt, \tag{3.2}$$

and

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)\ dt = \frac{1}{2}\int_{-\infty}^\infty e^{-z\cosh t - \nu t}\ dt. \tag{3.3}$$

We present our results below.

Theorem 3.1 (**Exp-arc series for Y_ν**). *Let $z, \nu \in \mathbb{C} - \{0\}$ with $\mathrm{Re}(z) > 0$, and let $N \in \mathbb{N}$. Define*

$$S(N,z,\nu) := \sum_{n=0}^{\infty}\left(\alpha_n(z)a_n(0,\nu) + \beta_n(z)\sum_{k=1}^{N} e^{-kz}a_n(k,\nu)\right) + \sum_{n=0}^{\infty} A_n(\nu)I_n(N+\tfrac{1}{2},z,\nu). \tag{3.4}$$

Then we have

$$Y_\nu(z) = \frac{1}{2\pi i}\left(e^{-i\nu\pi/2}\mathcal{I}(iz,\nu) - e^{i\nu\pi/2}\mathcal{I}(-iz,\nu)\right) + \frac{z}{\nu\pi}\left(\frac{1-\cos(\nu\pi)}{z} + S(N,z,\nu)\cos(\nu\pi) - S(N,z,-\nu)\right). \tag{3.5}$$

where $\mathcal{I}(p,q)$, $a_n(k,\nu)$, $A_n(\nu)$, $\alpha_n(z)$, $\beta_n(z)$, and $I_n(\Theta,z,\nu)$ are as in Theorem 2.4. When $\nu = 0$, we have

$$Y_0(z) = \frac{1}{2\pi i}\left(\mathcal{I}(iz,0) - \mathcal{I}(-iz,0)\right) + 2\sum_{n=0}^{\infty}\left(\alpha_{2n}(z)a^*_{2n}(0) + \beta_n(z)\sum_{k=1}^{N} e^{-kz}a^*_n(k) + a^*_{2n}(0)I_n(N+\tfrac{1}{2},z,1)\right), \tag{3.6}$$

where $a^*_n(k)$ satisfy

$$a^*_{n+1}(k) = -\frac{k(2n+1)a^*_n(k) + na^*_{n-1}(k)}{(k^2+1)(n+1)},$$

with $a^*_0(k) = (1+k^2)^{-1/2}$ and $a^*_1(k) = ka^*_0/(1+k^2)$.

Remark 3.2. One should note that when ν is a positive integer, the sum

$$\sum_{n=0}^{\infty} A_n(-\nu)I_n(N+\tfrac{1}{2},z,-\nu)$$

in $S(N,z,-\nu)$ may be written as

$$\sum_{n=0}^{\nu} A_n(-\nu)I_n(N+\tfrac{1}{2},z,-\nu) + \sum_{n=0}^{\infty}(-1)^{\nu+1}A_n(\nu)I_n(N+\tfrac{1}{2},z,\nu),$$

the infinite part of which cancels with the analogous sum in $S(N,z,\nu)$.

Proof of Theorem 3.1. The theorem follows immediately from (3.1) and the proof of Theorem 2.4 in the case where $\nu \neq 0$. For Y_0, note that the infinite integral becomes

$$2\int_0^\infty e^{-z\sinh t}dt = 2\int_0^\infty \frac{e^{-zs}}{\sqrt{1+s^2}}\,ds.$$

If we set

$$\frac{1}{\sqrt{1+(k+s)^2}} = \sum_{n=0}^{\infty} a^*_n(k)s^n,$$

then since

$$\frac{d}{ds}\frac{1}{\sqrt{1+(k+s)^2}} = -\frac{k+s}{1+(k+s)^2}\frac{1}{\sqrt{1+(k+s)^2}},$$

we find that

$$(k^2+1)(n+1)a^*_{n+1} + 2kna^*_n + (n-1)a^*_{n-1} = -ka^*_n - a^*_{n-1}.$$

Thus

$$a^*_{n+1} = -\frac{k(2n+1)a^*_n + na^*_{n-1}}{(k^2+1)(n+1)}$$

with $a^*_0 = (1+k^2)^{-1/2}$ and $a^*_1 = ka^*_0/(1+k^2)$. Note also that $a_{2n+1}(0) = 0$ and

$$\frac{1}{\sqrt{1+s^2}} = \frac{1}{s\sqrt{1+s^{-2}}} = \frac{1}{s}\sum_{n=0}^{\infty}\frac{a^*_n(0)}{s^n}$$

for $|s|>1$. Therefore, for any $N\in\mathbb{N}$ we may write

$$\begin{aligned}\int_0^\infty \frac{e^{-zs}}{\sqrt{1+s^2}}\,ds &= \left(\int_0^{1/2}+\sum_{k=1}^N e^{-ks}\int_{k-1/2}^{k+1/2}+\int_{N+1/2}^\infty\right)\frac{e^{-zs}}{\sqrt{1+s^2}}\,ds\\ &=\sum_{n=0}^\infty\left(a_{2n}^*(0)\alpha_{2n}(z)+\sum_{k=1}^N e^{-kz}a_n^*(k)\beta_n(z)+a_{2n}^*(0)\int_{N+1/2}^\infty e^{-zs}s^{-2n-1}ds\right)\\ &=\sum_{n=0}^\infty\left(a_{2n}^*(0)\alpha_{2n}(z)+\sum_{k=1}^N e^{-kz}a_n^*(k)\beta_n(z)+a_{2n}^*(0)I_n(N+\tfrac12,z,1)\right),\end{aligned}$$

which, when combined with (3.1) and the proof of Theorem 2.4, proves (3.6). $\square$

Theorem 3.3 (**Exp-arc series for I_ν and K_ν**). *Under the same conditions as for Theorem 3.1, define*

$$\mathcal{I}^*(z,\nu)=\frac{2e^z}{\nu}\sum_{n=0}^\infty\frac{r_{2n+2}(2i\nu)}{(2n+1)!}B_{n+\frac12}(z),\tag{3.7}$$

and

$$\begin{aligned}T(N,z,\nu):=\sum_{n=0}^\infty&\left(\frac{2e^{-z}}{2^{n/2}}a_n(0,2\nu)B_{\frac{n+1}{2}}(z/2)+\beta_n(-z)\sum_{k=2}^N e^{-kz}b_n(k,\nu)\right)\\ &+\sum_{n=0}^\infty(-1)^nA_n(\nu)I_n(N+\tfrac12,z,\nu).\end{aligned}\tag{3.8}$$

where $a_n(k,\nu)$, $\beta_n(z)$, $A_n(\nu)$, and $I_n(\Theta,z,\nu)$ are as in Theorem 2.4; $b_n(k,\nu)$ satisfy

$$b_{n+2}(k,\nu)=\frac{1}{k^2-1}\left(\frac{(\nu^2-n^2)b_n(k,\nu)+k(n+1)(2n+1)b_{n+1}(k,\nu)}{(n+2)(n+1)}\right)\tag{3.9}$$

with $b_0(k,\nu)=(k+\sqrt{k^2-1})^{-\nu}$ and $b_1(k,\nu)=\dfrac{\nu b_0}{\sqrt{k^2-1}}$; $r_k(\nu)$ is given by (2.3); and $B_k(p)$ is given by (2.6). Then we have

$$\begin{aligned}I_\nu(z)=&\frac{1}{2\pi}\left(\mathcal{I}(z,\nu)+\cos(\nu\pi)\mathcal{I}(-z,\nu)-\sin(\nu\pi)\mathcal{I}^*(-z,\nu)\right)\\ &+\frac{z\sin(\nu\pi)}{\nu\pi}\left(-\frac{e^{-z}}{z}+T(N,z,\nu)\right),\end{aligned}\tag{3.10}$$

and

$$K_\nu(z)=\frac{z}{2\nu}\left(T(N,z,-\nu)-T(N,z,\nu)\right),\tag{3.11}$$

where $\mathcal{I}(p,q)$ is given by (2.5). When $\nu=0$, we have

$$I_0(z)=\frac{1}{2\pi}\left(\mathcal{I}(z,0)+\mathcal{I}(-z,0)\right),\tag{3.12}$$

and

$$K_0(z)=\sum_{n=0}^\infty\left(\sqrt{2}e^{-z}d_n^*B_n(z/2)+\beta_n(-z)\sum_{k=2}^N e^{-kz}b_n^*(k)+(-1)^na_{2n}^*(0)I_n(N+\tfrac12,z,1)\right),\tag{3.13}$$

where

$$d_n^* := 2^{-n}\binom{-1/2}{n} = \frac{(-n+1/2)}{2n} d_{n-1}^*,$$

and the $b_n^*(k)$ satisfy

$$b_{n+1}^*(k) = \frac{k(2n+1)b_n^*(k) - nb_{n-1}^*(k)}{(k^2-1)(n+1)},$$

with $b_0^* = (k^2-1)^{-1/2}$ and $b_1^* = \dfrac{kb_0^*}{k^2-1}$, while $a_{2n}^*(0)$ is the same as in Theorem 3.1.

Remark 3.4. The reader may now be tempted to compare our series for $I_\nu(z)$ with Paris' formula (1.10). Indeed, for $\nu = m \in \mathbb{Z}$, our series looks like

$$I_\nu(z) = \frac{1}{2\pi}(I(z,m) + (-1)^m \mathcal{I}(-z.m)),$$

and both the summands of $\mathcal{I}$ as well as those of (1.10) have the form

$$\frac{(rising\ factorial) \times (incomplete\ gamma)}{power\ of\ z}.$$

However, there are several key differences between the exp-arc series and (1.10):

- Although the summands in both the exp-arc and Paris' series involve incomplete gamma functions that are recursively computable, Paris' formula involves $\gamma(k+\nu+\frac{1}{2}, \pm z)$ while our series only involve $B_k(\pm z)$, which can be expressed in terms of $\gamma(\frac{1}{2}, \pm z)$. That is, our incomplete gamma evaluations are independent of ν.
- When ν is not an integer, our series adjusts for the above independence with the sums $T(N,z,\nu)$. These sums involve $\beta_n(z)$ which, although technically is an incomplete gamma function, is explicitly computable in closed form since it is the nth moment of the exponential. Thus, the only integral with ν dependence is $I_n(N+\frac{1}{2}, z, \nu)$, which, as we will see in the next section, can be ignored if one chooses a large enough N.

Proof. Since the proof is very similar to that of Theorems 2.4 and 3.1, we only highlight the differences here and refer the reader to the Appendix for the details. Note that the integral on $[0,\pi]$ in (3.2) simplifies to

$$\begin{aligned}\int_0^\pi e^{z\cos t}\cos(\nu t)\ dt &= \frac{1}{2}\mathcal{I}(z,\nu) + \int_{\pi/2}^\pi e^{z\cos t}\cos(\nu t)\ dt \\ &= \frac{1}{2}\left(\mathcal{I}(z,\nu) + e^{i\nu\pi}\mathcal{I}(-z,\nu,0,\pi/2) + e^{-i\nu\pi}\mathcal{I}(-z,-\nu,0,\pi/2)\right).\end{aligned}$$

Using (2.4) and combining the even (resp. odd) index terms into one sum, we obtain

$$\int_0^\pi e^{z\cos t}\cos(\nu t)\,dt = \frac{1}{2}\left(\mathcal{I}(z,\nu) + \cos(\nu\pi)\mathcal{I}(-z,\nu) - \sin(\nu\pi)\sum_{n=0}^\infty \frac{2e^{-z}r_{2n+2}(2i\nu)}{(2n+1)!\nu} B_{n+\frac{1}{2}}(-z)\right).$$

Turning to the infinite integrals, after an integration by parts for $\nu \neq 0$ as in the previous theorems, it suffices to show that

$$\int_1^\infty e^{-zs}e^{-\nu\,\mathrm{arccosh}\,s}ds = T(N,z,\nu)$$

for every $N \in \mathbb{N}$. From the second order differential equation satisfied by $e^{-\nu \operatorname{arccosh} s}$, it is easy to see that, for $k \geq 2$,

$$e^{-\nu \operatorname{arccosh}(k-s)} = \sum_{n=0}^{\infty} b_n(k,\nu) s^n,$$

where $b_n(k,\nu)$ are given by (3.9). It is also easy to verify that

$$s^{\nu} e^{-\nu \operatorname{arccosh} s} = \sum_{n=0}^{\infty} \frac{(-1)^n A_n(\nu)}{s^{2n}}.$$

Thus, applying these expansions and interchanging summation and integration, we find

$$\int_{3/2}^{\infty} e^{-zs} e^{-\nu \operatorname{arccosh} s} ds = \sum_{k=2}^{N} e^{-kz} \sum_{n=0}^{\infty} b_n(k,\nu)\beta_n(-z) + \sum_{n=0}^{\infty} (-1)^n A_n(\nu) I_n(N+\tfrac{3}{2}, z, \nu).$$

Now all that remains is to show that

$$\int_{1}^{3/2} e^{-zs} e^{-\nu \operatorname{arccosh} s} ds = \sum_{n=0}^{\infty} \frac{2e^{-z}}{2^{n/2}} a_n(0,2\nu) B_{\frac{n+1}{2}}(z/2).$$

To do this, we note that $e^{-\nu \operatorname{arccosh} s}$ does not have a power series at the point $s=1$. However, if we set $u = \sqrt{s-1}$, then it can be verified that

$$e^{-\nu \operatorname{arccosh} s} = \sum_{n=0}^{\infty} a_n(0,2\nu) u^n,$$

and so

$$\begin{aligned}\int_{1}^{3/2} e^{-zs} e^{-\nu \operatorname{arccosh} s} ds &= \int_{0}^{1/\sqrt{2}} e^{-z(u^2+1)} \sum_{n=0}^{\infty} a_n(0,2\nu) u^n 2u du \\ &= 2e^{-z} \sum_{n=0}^{\infty} 2^{-n/2} a_n(0,2\nu) B_{(n+1)/2}(z/2).,\end{aligned}$$

This completes the proof for the case where $\nu \neq 0$. In the case where $\nu = 0$, we now need only to evaluate K_0 and so we may follow the approach for the proof of (3.6). Expanding the function $(1+s^2)^{-1/2}$ at each integer $k \geq 2$ as well as at ∞ gives us the sums involving b_n^* and A_n in (3.13), while expanding $(s+1)^{-1/2}$ as a series in $u = \sqrt{s-1}$ gives the sum involving d_n^*. □

4. Effectiveness of these Series

We now turn our attention to the performance of our series in Theorems 2.4, 3.1, and 3.3. These series can be separated into three parts: a pair of sums from the $\mathcal{I}$ function, a number of sums involving the moments of the exponential (given by α_n and β_n), and a series involving the incomplete gamma function arising from the tail of the infinite integrals. Let us consider each of these separately. First, we look at the rate of convergence for the $\mathcal{I}$ sums.

Our series for J_ν, Y_ν, and I_ν involve terms of the form $\mathcal{I}(i^k z, \nu)$, which by Theorem 2.1 can be expressed as

$$\mathcal{I}(i^k z, \nu) = 4e^{i^k z} \sum_{n=0}^{\infty} c_n(\nu) B_n(i^k z),$$

where

$$c_n(\nu) := \frac{1}{(2n)!}\prod_{j=1}^{n}\left((2j-1)^2-4\nu^2\right) = \prod_{j=1}^{n}\left(1-\frac{1}{2j}-\frac{4\nu^2}{(2j-1)(2j)}\right),$$

with $c_0 := 1$, and

$$B_n(i^k z) = \frac{1}{2^{n+3/2}}\int_0^1 e^{-i^k z u}u^{n-1/2}du.$$

Thus it is clear that $c_n(\nu)$ is bounded for fixed ν. In fact, it is easy to see that $|c_n(\nu)|$ is strictly decreasing for $n \geq 2|\nu|^2 - 1/2$. It is also clear that for all $n \geq 1$ we have

$$|B_n(i^k z)| \leq \frac{\max(1, e^{-\operatorname{Re}(i^k z)})}{2^{n+3/2}}.$$

Therefore, for fixed k the error when the $\mathcal{I}(i^k z, \nu)$ sum is truncated after M terms can be bounded by

$$\begin{aligned}\left|4e^{i^k z}\sum_{n\geq M+1} c_n(\nu)B_n(i^k z)\right| &\leq |4e^{i^k z}|\sum_{n\geq M+1}\frac{C(\nu)\max(1, e^{-\operatorname{Re}(i^k z)})}{2^{n+3/2}} \\ &= \frac{4C(\nu)}{2^{M+3/2}}\cdot\max(e^{\operatorname{Re}(i^k z)}, 1) \qquad (4.1)\end{aligned}$$

for some constant $C(\nu)$ depending only on ν.

Our series for I_ν also contains terms of the form $\mathcal{I}^*(-z,\nu)$. By a similar argument as above, there exists a constant $C^*(\nu)$ such that the tail of $\mathcal{I}^*(-z,\nu)$ when truncated after M terms is bounded by

$$\begin{aligned}\left|\frac{2e^{-z}}{\nu}\sum_{n\geq M+1}\frac{r_{2n+2}(2i\nu)}{(2n+1)!}B_{n+1/2}(-z)\right| &\leq |2e^{-z}|\sum_{n\geq M+1}\frac{C^*(\nu)\max(1, e^{\operatorname{Re}(z)})}{2^{n+2}} \\ &= \frac{C^*(\nu)}{2^{M+1}}\cdot\max(e^{-\operatorname{Re}(z)}, 1). \qquad (4.2)\end{aligned}$$

Now, let us consider the sums arising from the main contribution from the infinite exp-arc integrals. In the case of the J and Y Bessel functions, these sums are of the form $\sum a_n(0,\nu)\alpha_n(z)$ and $e^{-kz}\sum a_n(k,\nu)\beta_n(z)$, and in the case of the I and K Bessel functions, they are of the form $e^{-z}\sum 2^{1-n/2}a_n(0,2\nu)B_{(n+1)/2}(z/2)$ and $e^{-kz}\sum b_n(k,\nu)\beta_n(-z)$. By (2.11), we deduce that, for $n \geq 1$,

$$a_{2n}(0,\nu) = \frac{1}{(2n)!}\prod_{k=0}^{n-1}(\nu^2-(2k)^2) = \frac{(-1)^{n-1}\nu^2}{2n}\prod_{k=2}^{n}\left(1-\frac{1}{2k-1}-\frac{\nu^2}{(2k-1)(2k-2)}\right)$$

and

$$a_{2n+1}(0,\nu) = -\frac{\nu}{(2n+1)!}\prod_{k=0}^{n-1}(\nu^2-(2k+1)^2) = \frac{(-1)^{n+1}\nu}{2n+1}c_n(\nu/2).$$

Thus, we may conclude that for $n > |\nu|^2/2$, $|na_n(0,\nu)|$ is decreasing, and that for all $n \geq 2$ there is a constant $C_0^*(\nu)$ such that

$$|a_n(0,\nu)| \leq \frac{C_0^*(\nu)}{n}. \qquad (4.3)$$

For the more general sequences $a_n(k,\nu)$ with $k \geq 1$, since the series $\sum a_n(k,\nu)s^n$ has radius of convergence $\sqrt{k^2+1}$, for sufficiently large n on using the root-test we have $|a_n(k,\nu)| <$

$(k^2+1)^{-n/2}$. Similarly, for each $k > 2$ we have $|b_n(k,\nu)| < (k-1)^{-n}$ for sufficiently large n. We may make explicit the meaning of "sufficiently large"—at the expense of somewhat worse bounds—by using the recurrence relations (2.11) and (3.9). Note that when $n > \frac{1}{3}(|\nu|^2+4)$, we have $\left|\frac{\nu^2-(n-2)^2}{n(n-1)}\right| < 1$. Thus in this range of n, we have

$$|a_n(k,\nu)| < \frac{2k+1}{k^2+1}\max(|a_{n-2}(k,\nu)|, |a_{n-1}(k,\nu)|),$$

and

$$|b_n(k,\nu)| < \frac{2k+1}{k^2-1}\max(|b_{n-2}(k,\nu)|, |b_{n-1}(k,\nu)|).$$

We may conclude that there exist effectively computable constants $C_k^*(\nu)$ and $D_k^*(\nu)$ such that when $n > \frac{1}{3}(|\nu|^2+4)$ we have

$$|a_n(k,\nu)| < \begin{cases} \left(\dfrac{3}{2}\right)^n C_1^*(\nu), & \text{for } k=1, \\ \\ \left(\dfrac{2k+1}{k^2+1}\right)^{n/2} C_k^*(\nu), & \text{for } k>1, \end{cases}$$

and

$$|b_n(k,\nu)| < \begin{cases} \left(\dfrac{5}{3}\right)^n D_2^*(\nu), & \text{for } k=2, \\ \\ \left(\dfrac{2k+1}{k^2-1}\right)^{n/2} D_k^*(\nu), & \text{for } k>2. \end{cases}$$

Note that we may choose the constants such that these inequalities hold for all $n > 0$. We bound $|\alpha_n(z)|$ and $|\beta_n(z)|$ trivially in the range $\mathrm{Re}(z) > 0$, so that for all $n > 0$,

$$|\alpha_n(z)| < \frac{1}{2^{n+1}} \qquad \text{and} \qquad |\beta_n(\pm z)| < \frac{e^{\mathrm{Re}(z)/2}}{2^n}.$$

Finally, we turn our attention to the series arising from the tails of the infinite integrals: that is, the sums

$$\sum_{n=0}^{\infty}(\pm 1)^n A_n(\nu) I_n(N+\tfrac{1}{2}, z, \pm\nu).$$

Now, by (2.15) we have

$$|A_n| \le \frac{|\nu 2^{-\nu}|}{n2^{2n}}\binom{V+2n-1}{n-1} < \frac{|\nu 2^{-\nu+V+2n-1}|}{n2^{2n}} = \frac{|\nu 2^{V-\nu-1}|}{n},$$

where $V := \lceil |\nu| \rceil$, the smallest integer greater than or equal to $|\nu|$. Also, when $\mathrm{Re}(z) > 0$, we may trivially bound $I_n(N+\frac{1}{2}, z, \nu)$ by

$$|I_n(N+\tfrac{1}{2}, z, \nu)| \le \frac{e^{-(N+\frac{1}{2})\mathrm{Re}(z)}}{(N+\frac{1}{2})^{2n+\mathrm{Re}(\nu)-1}} \int_0^\infty e^{-(N+\frac{1}{2})\mathrm{Re}(z)s}(1+s)^{-2n-\mathrm{Re}(\nu)}ds,$$

which is bounded and monotonically decreasing as n increases. We can obtain a simpler bound when $2n > \mathrm{Re}(\nu)$, since by [2, Thm 2.3] we have

$$\left|\int_0^\infty e^{-zs}(1+s)^{a-1}ds\right| \le \frac{1}{|z|}\left(1 + \left|\frac{1-a}{1-\mathrm{Re}(a)}\right|\right)$$

whenever $\mathrm{Re}(a-1)<0$. Thus, we find that, whenever $2n>\mathrm{Re}(\nu)$,

$$\begin{aligned}|I_n(N+\tfrac{1}{2},z,\nu)| &\leq \frac{e^{-(N+\frac{1}{2})\mathrm{Re}(z)}}{(N+\frac{1}{2})^{2n+\mathrm{Re}(\nu)-1}}\frac{1}{|(N+\frac{1}{2})z|}\left(1+\left|\frac{2n+\nu}{2n+\mathrm{Re}(\nu)}\right|\right)\\ &\leq \left(\frac{2+|\frac{\mathrm{Im}(\nu)}{2n+\mathrm{Re}(\nu)}|}{(N+\frac{1}{2})^{\mathrm{Re}(\nu)}}\right)\frac{e^{-(N+\frac{1}{2})\mathrm{Re}(z)}}{|z|}\frac{1}{(N+\frac{1}{2})^{2n}}.\end{aligned}$$

We summarize our discussion in the following theorem.

Theorem 4.1. *Let z, $\nu\in\mathbb{C}$ with $\mathrm{Re}(z)>0$. Then we obtain that:*

(1) *For each positive integer k there exist effectively computable constants $C(\nu)$, $C^*(\nu)$, $C_k^*(\nu)$, $D_k^*(\nu)$ such that for any positive integer M,*

$$\left|4e^{i^kz}\sum_{n>M}c_n(\nu)B_n(i^kz)\right|\leq\frac{4C(\nu)}{2^{M+3/2}}\cdot\max(e^{\mathrm{Re}(i^kz)},1),\tag{4.4}$$

$$\left|\frac{2e^{-z}}{\nu}\sum_{n>M}\frac{r_{2n+2}(2i\nu)}{(2n+1)!}B_{n+1/2}(-z)\right|=\frac{C^*(\nu)}{2^{M+1}}\cdot\max(e^{-\mathrm{Re}(z)},1).\tag{4.5}$$

$$\left|\sum_{n>M}a_n(0,\nu)\alpha_n(z)\right|\leq\frac{C_0^*(\nu)}{M2^{M+1}},\tag{4.6}$$

$$\left|e^{-z}\sum_{n>M}a_n(1,\nu)\beta_n(z)\right|\leq 3C_1^*(\nu)e^{-\mathrm{Re}(z)/2}\left(\frac{3}{4}\right)^M\tag{4.7}$$

$$\begin{aligned}\left|e^{-kz}\sum_{n>M}a_n(k,\nu)\beta_n(z)\right|&\leq C_k^*(\nu)e^{-(k-\frac{1}{2})\mathrm{Re}(z)}\\ &\quad\times\left(\frac{1}{2}\sqrt{\frac{2k+1}{k^2+1}}\right)^{M+1}\left(1-\frac{1}{2}\sqrt{\frac{2k+1}{k^2+1}}\right)^{-1},\end{aligned}\tag{4.8}$$

$$\left|2e^{-z}\sum_{n>M}\frac{a_n(0,2\nu)}{2^{n/2}}B_{(n+1)/2}(z/2)\right|\leq\frac{C_0^*(2\nu)e^{-\mathrm{Re}(z)}}{M2^{M+1}},\tag{4.9}$$

$$\left|e^{-z}\sum_{n>M}b_n(2,\nu)\beta_n(-z)\right|\leq 5D_2^*(\nu)e^{-\mathrm{Re}(z)/2}\left(\frac{5}{6}\right)^M\tag{4.10}$$

$$\begin{aligned}\left|e^{-kz}\sum_{n>M}b_n(k,\nu)\beta_n(-z)\right|&\leq D_k^*(\nu)e^{-(k-\frac{1}{2})\mathrm{Re}(z)}\\ &\quad\times\left(\frac{1}{2}\sqrt{\frac{2k+1}{k^2-1}}\right)^{M+1}\left(1-\frac{1}{2}\sqrt{\frac{2k+1}{k^2-1}}\right)^{-1}.\end{aligned}\tag{4.11}$$

We also obtain that:

(2) *For any positive integer N there exists an effectively computable constant $C_\infty(\nu)$ such that for any positive integer M*

$$\left|\sum_{n>M}(\pm1)^nA_n(\nu)I_n(N+\tfrac{1}{2},z,\pm\nu)\right|\leq\frac{C_\infty(\nu)e^{-(N+\frac{1}{2})\mathrm{Re}(z)}}{|z|}$$

$$\times \frac{1}{M(N+\frac{1}{2})^{2(M+1)}} \left(1 - \frac{1}{(N+\frac{1}{2})^2}\right)^{-1}. \tag{4.12}$$

If, moreover, $2M > \mathrm{Re}(\nu)+1$ *then we have the bound*

$$|C_\infty(\nu)| \leq \frac{|\nu 2^{|\nu|}|(2+|\mathrm{Im}(\nu)|)}{(2N+1)^{\mathrm{Re}(\nu)}}.$$

From Theorem 4.1 we can deduce the following bounds for the errors in computing Bessel functions using the series in Theorems 2.4, 3.1, and 3.3. For simplicity of illustration, in the next corollary we set $N=1$ in each of the theorems, and truncate each infinite series at $n=M$.

Corollary 4.2. *Let* $z,\nu \in \mathbb{C}$ *with* $\mathrm{Re}(z) > 0$*. For each integer* $M > 0$*, set*

$$\begin{aligned}
\mathcal{I}_M(z,\nu) &:= 4e^z \sum_{n=0}^{M} c_n(\nu) B_n(z),\\
\mathcal{I}^*_M(z,\nu) &:= 2e^z \sum_{n=0}^{M} \frac{r_{2n+2}(2i\nu)}{(2n+1)!\nu} B_{n+\frac{1}{2}}(z),\\
S^*(M,z,\nu) &:= \sum_{n=0}^{M} \left(\alpha_n(z) a_n(0,\nu) + e^{-z}\beta_n(z) a_n(1,\nu) + A_n(\nu) I_n(\tfrac{3}{2},z,\nu)\right),\\
T^*(M,z,\nu) &:= \sum_{n=0}^{M} \left(\frac{2e^{-z}}{2^{n/2}} a_n(0,2\nu) B_{\frac{n+1}{2}}(z/2) + (-1)^n A_n(\nu) I_n(\tfrac{3}{2},z,\nu)\right),\\
C_\infty(\nu) &:= \left|\frac{\nu 2^{|\nu|}(2+|\mathrm{Im}(\nu)|)}{3^{\mathrm{Re}(\nu)}}\right|.
\end{aligned} \tag{4.13}$$

Then for any M *with* $2M > |\mathrm{Re}(\nu)|+1$*, the* errors E_J*,* E_Y*,* E_I*, and* E_K *from the truncation of the exp-arc series after* M *terms at* $N=1$ *defined by*

$$\begin{aligned}
E_J(M,z,\nu) &:= J_\nu(z) - \frac{1}{2\pi}\left(e^{-i\nu\pi/2}\mathcal{I}_M(iz,\nu) + e^{i\nu\pi/2}\mathcal{I}_M(-iz,\nu)\right)\\
&\quad - \frac{z\sin(\nu\pi)}{\nu\pi}\left(-\frac{1}{z} + S^*(M,z,\nu)\right),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
E_Y(M,z,\nu) &:= Y_\nu(z) - \frac{1}{2\pi i}\left(e^{-i\nu\pi/2}\mathcal{I}_M(iz,\nu) - e^{i\nu\pi/2}\mathcal{I}_M(-iz,\nu)\right)\\
&\quad - \frac{z}{\nu\pi}\left(\frac{1-\cos(\nu\pi)}{z} + S^*(M,z,\nu)\cos(\nu\pi) - S^*(M,z,-\nu)\right),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
E_I(M,z,\nu) &:= I_\nu(z) - \frac{1}{2\pi}\left(\mathcal{I}_M(z,\nu) + e^{\pi i\nu}\mathcal{I}^*_M(-z,\nu) + e^{-\pi i\nu}\mathcal{I}^*_M(-z,-\nu)\right)\\
&\quad - \frac{z\sin(\nu\pi)}{\nu\pi}\left(-\frac{e^{-z}}{z} + T^*(M,z,\nu)\right),
\end{aligned} \tag{4.16}$$

$$E_K(M,z,\nu) := K_\nu(z) - \frac{z}{2\nu}\left(T^*(M,z,-\nu) - T^*(M,z,\nu)\right), \tag{4.17}$$

are bounded by

$$
\begin{aligned}
|E_J(M,z,\nu)| \le{}& \frac{C(\nu)}{2^{M+1/2}\pi}\left(e^{\operatorname{Im}(\nu)\pi/2}\max(e^{-\operatorname{Im}(z)},1)+e^{-\operatorname{Im}(\nu)\pi/2}\max(e^{\operatorname{Im}(z)},1)\right)\\
&+\left|\frac{z\sin(\nu\pi)}{\nu\pi}\right|\left(\frac{C_0^*(\nu)}{M2^{M+1}}+\frac{3C_1^*(\nu)e^{-\operatorname{Re}(z)/2}}{(4/3)^M}+\frac{5}{9}\frac{C_\infty(\nu)e^{-3\operatorname{Re}(z)/2}}{M(9/4)^{M+1}|z|}\right),
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
|E_Y(M,z,\nu)| \le{}& \frac{C(\nu)}{2^{M+1/2}\pi}\left(e^{\operatorname{Im}(\nu)\pi/2}\max(e^{-\operatorname{Im}(z)},1)+e^{-\operatorname{Im}(\nu)\pi/2}\max(e^{\operatorname{Im}(z)},1)\right)\\
&+\left|\frac{z}{\nu\pi}\right|\left(\frac{C_0^*(\nu)|\cos(\nu\pi)|+C_0^*(-\nu)}{M2^{M+1}}\right.\\
&\qquad+\frac{3\left(C_1^*(\nu)|\cos(\nu\pi)|+C_1^*(-\nu)\right)e^{-\operatorname{Re}(z)/2}}{(4/3)^M}\\
&\qquad\qquad\left.+\frac{5}{9}\frac{\left(C_\infty(\nu)|\cos(\nu\pi)|+C_\infty(-\nu)\right)e^{-3\operatorname{Re}(z)/2}}{M(9/4)^{M+1}|z|}\right),
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
|E_I(M,z,\nu)| \le{}& \frac{C(\nu)e^{\operatorname{Re}(z)}+C^*(\nu)|\cos(\nu\pi)|}{2^{M+1/2}\pi}+\frac{C^*(-\nu)|\sin(\nu\pi)|}{2^{M+2}\pi}\\
&+\left|\frac{z\sin(\nu\pi)}{\nu\pi}\right|\left(\frac{C_0^*(2\nu)e^{-\operatorname{Re}(z)}}{M2^{M+1}}+\frac{5}{9}\frac{C_\infty(\nu)e^{-3\operatorname{Re}(z)/2}}{M(9/4)^{M+1}|z|}\right),
\end{aligned}
\tag{4.20}
$$

and

$$
|E_K(M,z,\nu)| \le \left|\frac{z}{2\nu}\right|\left(\frac{\left(C_0^*(2\nu)+C_0^*(-2\nu)\right)e^{-\operatorname{Re}(z)}}{M2^{M+1}}+\frac{5\left(C_\infty(\nu)+C_\infty(-\nu)\right)e^{-3\operatorname{Re}(z)/2}}{9M(9/4)^{M+1}|z|}\right).
\tag{4.21}
$$

In consequence, as M tends to infinity, we have

$$
|E(M,z,\nu)| = O_{\nu,z}\left(\frac{1}{2^M}\right),
\tag{4.22}
$$

where $E(M,z,\nu)$ denotes any of the functions E_J, E_Y, E_I, or E_K, and the notation $O_{\nu,z}$ indicates that the big-O constant depends on both ν and z.

Proof. As given above, the formulas for J, Y, I, and K are simply restatements of Theorems 2.4, 3.1, and 3.3 with $N = 1$ and truncation at M terms. The bounds for the errors follow immediately upon application of Theorem 4.1. Specifically, (4.18) and (4.19) follow from (4.4), (4.6), (4.7), and (4.12); (4.20) follows from (4.4), (4.9), and (4.12); and (4.21) follows from (4.9) and (4.12).

The asymptotic (4.22) for the error is easily deduced from the fact that $a_n(1,\nu) = O(2^{-n/2})$ as n tends to infinity, so that the $(4/3)^M$ in (4.18) and (4.19) may be replaced by 2^M. □

We close this section with a few notes on the implementation of the series.

4.1. **Notes on Implementation.**

1. First, we make a few remarks on the choice of N and the implementation of the S and T sums. Actually, we limit our discussion to $S(N,\nu,z)$, since the case $T(N,\nu,z)$ is similar. For a fixed N, we have $N+1$ infinite sums of the form $S_k := e^{-kz}\sum_n a_n(k,\nu)\beta_n(z)$, $0 \le k \le N$, and a sum $S_\infty := \sum_n A_n I_n$ for the tail of the infinite integral. Note that

for each k, the size of S_k is of the order $O(e^{-(k-1/2)z})$, and that S_∞ is of the order $O(e^{-(N+1/2)z})$. The O-constants are explicitly computable by Theorem 4.1, and thus it is possible to determine at what point it becomes necessary to begin summing the terms of S_k. Note also that as k increases, the rate of convergence of S_k also increases, and fewer terms are needed. As well, if one chooses a large enough N, it is possible to entirely avoid the error-function evaluation that is necessary in computing S_∞.

(2) Second, we have stated our theorems for $\mathrm{Re}(z) > 0$. To evaluate the Bessel functions when $\mathrm{Re}(z) < 0$, one should use the well-known formulas [3, Secs. 10.11, 10.34],

$$J_\nu(ze^{m\pi i}) = e^{m\nu\pi i} J_\nu(z), \qquad I_\nu(ze^{m\pi i}) = e^{m\nu\pi i} I_\nu(z),$$

$$Y_\nu(ze^{m\pi i}) = e^{-m\nu\pi i} Y_\nu(z) + 2i\sin(m\nu\pi)\cot(\nu\pi) J_\nu(z),$$
$$K_\nu(ze^{m\pi i}) = e^{-m\nu\pi i} K_\nu(z) - \pi i\sin(m\nu\pi)\csc(\nu\pi) I_\nu(z),$$

where $m \in \mathbb{Z}$. For the details on how to choose m, see [14, p. 75]. Along the same lines, it is useful to use the identities [3, Sec. 10.27]

$$I_\nu(z) = e^{-\nu\pi i/2} J_\nu(iz),$$
$$-\pi i J_\nu(z) = e^{-\nu\pi i/2} K_\nu(-iz) - e^{\nu\pi i/2} K_\nu(iz),$$

and

$$-\pi Y_\nu(z) = e^{-\nu\pi i/2} K_\nu(-iz) + e^{\nu\pi i/2} K_\nu(iz),$$

to evaluate $J_\nu(z)$ and $Y_\nu(z)$ when $\mathrm{Im}(z) >> \mathrm{Re}(z)$.

(3) We also mention that in each of the sums $\mathcal{I}$ and S_k, the summands consist of a product of functions that depend only on ν and on functions that depend only on z. This facilitates one-ν many-z or one-z many-ν computations by allowing us to pre-compute either the coefficients $a_n(k,\nu)$ or the exponential moments $\beta_n(z)$. Note also that all of these coefficients are either bounded or are converging to zero—as opposed to the analogous functions found in the ascending series, where the dependence on z diverges to ∞, or those in the asymptotic series, where the dependence on ν diverges to ∞.

(4) Finally, we note that the integrals $B_k(p)$ and $I_n(\Theta, z, \nu)$ in our series (the terms that cannot be evaluated in closed form) are expressible in terms of the incomplete gamma function $\gamma(a,z)$ and its complement $\Gamma(a,z) := \Gamma(a) - \gamma(a,z)$. Since the incomplete gamma functions have elegant continued-fraction representations, our results translate to a continued-fraction computation scheme for the Bessel functions.

5. Numerical Results

To give a more realistic idea of the effectiveness of our theorems, we implemented Corollary 4.2 in Maple to compare it with the known ascending and asymptotic series. We remark that in addition to providing the numerical data given in Tables 1 and 2, implementation of our theorems have helped us troubleshoot the theorems themselves. Not only have we corrected minor sign errors and typos, the computations also alerted us to more fundamental issues such as the fact that recurrence (2.17) needs to be restarted at the index $n = -\nu$ if ν is a negative integer.

Table 1 shows the absolute difference between the true value of $J_\nu(z)$ and each of the values of (1.2) and (1.3) when truncated at M terms, as well as the absolute error $|E_J(M,z,\nu)|$ given by (4.14). In the case of (1.3), "truncated at M terms" means that both infinite sums are truncated at M terms. The data show that the exp-arc series converges like 2^{-M} as expected,

TABLE 1. Comparison between various series for $J_\nu(z)$.

		Absolute value of the difference between the true value and		
(ν, z)	M	Ascending Series (1.2)	Asymptotic Series (1.3)	Exp-arc Series (4.14)
	10	10^{4}	10^{-13}	10^{3}
$\nu = 12.3$	50	10^{-39}	10^{-5}	10^{-17}
$z = 20$	100	10^{-130}	10^{51}	10^{-33}
	150	10^{-242}	10^{130}	10^{-48}
	200	10^{-368}	10^{223}	10^{-64}
	10	10^{18}	10^{-23}	10^{2}
$\nu = 12.3$	30	10^{17}	10^{-41}	10^{-10}
$z = 50$	50	10^{6}	10^{-45}	10^{-17}
	100	10^{-45}	10^{-28}	10^{-33}
	150	10^{-117}	10^{11}	10^{-48}
	10	10^{27}	10^{-4}	10^{13}
$\nu = 12.3$	50	10^{38}	10^{-48}	10^{-17}
$z = 75 + 57i$	100	10^{14}	10^{-59}	10^{-33}
	120	10^{-2}	10^{-56}	10^{-39}
	150	10^{-31}	10^{-47}	10^{-48}
	200	10^{-89}	10^{-20}	10^{-64}

giving one good digit approximately every three terms. Similar results can be seen in Table 2, where we compare the performance of the various series for the modified Bessel function $I_\nu(z)$. Although it seems that in the $J_\nu(z)$ case the exp-arc series does not perform as well as either the ascending series (1.2) or the asymptotic series (1.3) in their respective domains of usefulness, both the exp-arc and Paris' series for $I_\nu(z)$ out-perform the ascending and Hadamard series for sufficiently large argument z. The data also indicate that the accuracy of the exp-arc series depends mainly on M, as the absolute errors do not seem to vary by much across values of z and ν. One should also note the following.

- For large z and small ν, to compute beyond the digital limit of the asymptotic series using the ascending series requires higher precision arithmetic because of the cancellation of large numbers from the initial terms. These terms are of size $\dfrac{(z/2)^k}{k!}\dfrac{(z/2)^{k+\nu}}{\Gamma(k+\nu+1)}$, whose maximum occurs near $k = z/2$ if $z >> \nu$. Compare this with the asymptotic $J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos(z - \frac{\pi\nu}{2} - \frac{\pi}{4})$.
- Switching from the asymptotic to the ascending series to obtain higher precision results requires that the computation using the asymptotic series be abandoned and the calculation restarted from scratch using the ascending series.
- It is not always obvious when to use the ascending series and when to use the asymptotic series, partly because it is not always obvious what amount of precision one requires before one begins a computation.
- To fairly compare the amount of work involved in our methods to that of Paris [4] or of the conventional ascending-asymptotic dichotomy depends on the context and on a level of implementation detail beyond the scope of the present article. That said, additional tabular comparison for the case of integer order is given in [1].

TABLE 2. Comparison between various series for $I_\nu(z)$.

		Absolute value of the difference between the true value and			
(ν, z)	M	Ascending (1.7)	Hadamard (1.9)	Paris (1.10)	Exp-arc (4.16)
$\nu = 4.2$, $z = 20$	10	10^{7}	10^{-3}	10^{-3}	10^{-3}
	50	10^{-33}	10^{-11}	10^{-19}	10^{-18}
	100	10^{-122}	10^{-12}	10^{-36}	10^{-33}
	150	10^{-232}	10^{-13}	10^{-52}	10^{-49}
	200	10^{-357}	10^{-14}	10^{-68}	10^{-64}
$\nu = 4.2$, $z = 50$	10	10^{21}	10^{7}	10^{7}	10^{7}
	30	10^{19}	10^{-9}	10^{-9}	10^{-9}
	50	10^{9}	10^{-17}	10^{-17}	10^{-17}
	100	10^{-40}	10^{-23}	10^{-34}	10^{-33}
	150	10^{-110}	10^{-24}	10^{-51}	10^{-49}
$\nu = 4.2$, $z = 75 + 57i$	10	10^{32}	10^{13}	10^{13}	10^{13}
	50	10^{39}	10^{-20}	10^{-17}	10^{-18}
	100	10^{17}	10^{-33}	10^{-33}	10^{-33}
	120	10^{2}	10^{-34}	10^{-40}	10^{-39}
	150	10^{-26}	10^{-34}	10^{-49}	10^{-49}
	200	10^{-84}	10^{-35}	10^{-65}	10^{-64}

6. Conclusion

The exp-arc expansion developed herein provides a geometrically convergent middle-ground between the asymptotic and ascending series that avoids the issues raised in the previous section. It provides a uniform approach to evaluating Bessel functions that is universally convergent, with *explicitly computable* error bounds. It is therefore easy to predict the number of terms needed in our expansion to guarantee a given number of correct digits.

Appendix

As promised we provide the details for the proof of Theorem 3.3 below.

Proof of Theorem 3.3. We wish to evaluate the integrals (3.2) and (3.3) as infinite series in the form of Theorem 3.3. As in the case for J_ν and Y_ν, we express the integral on $[0, \pi]$ in terms of $\mathcal{I}$. One easily finds that

$$\begin{aligned}\int_0^\pi e^{z\cos t}\cos(\nu t)\,dt &= \frac{1}{2}\mathcal{I}(z,\nu) + \int_{\pi/2}^\pi e^{z\cos t}\cos(\nu t)\,dt \\ &= \frac{1}{2}\left(\mathcal{I}(z,\nu) + e^{i\nu\pi}\mathcal{I}(-z,\nu,0,\pi/2) + e^{-i\nu\pi}\mathcal{I}(-z,-\nu,0,\pi/2)\right).\end{aligned}$$

Now, by (2.4) we may write

$$\begin{aligned}\mathcal{I}(-z,\pm\nu,0,\pi/2) &= \frac{ie^{-z}}{\pm\nu}\sum_{k=0}^\infty \frac{r_{k+1}(\mp 2i\nu)}{k!}B_{k/2}(-z) \\ &= \frac{ie^{-z}}{\pm\nu}\left(\sum_{k=0}^\infty \frac{r_{2k+1}(\mp 2i\nu)}{(2k)!}B_k(-z) + \frac{r_{2k+2}(\mp 2i\nu)}{(2k+1)!}B_{k+1/2}(-z)\right).\end{aligned}$$

Since r_{2k+1} is an odd function of ν and r_{2k+2} is an even function of ν, we find that

$$\mathcal{I}(-z,\pm\nu,0,\pi/2)=\frac{ie^{-z}}{\nu}\left(\sum_{k=0}^{\infty}\frac{r_{2k+1}(2i\nu)}{(2k)!}B_k(-z)\pm\frac{r_{2k+2}(2i\nu)}{(2k+1)!}B_{k+1/2}(-z)\right).$$

Thus, combining this with the expression for $\mathcal{I}(z,\nu)$ in (2.5), we have

$$\begin{aligned}&e^{\pi i\nu}\mathcal{I}(-z,\nu,0,\pi/2)+e^{-\pi i\nu}\mathcal{I}(-z,-\nu,0,\pi/2)\\&\qquad=\cos(\nu\pi)\mathcal{I}(-z,\nu)-2\sin(\nu\pi)\frac{e^{-z}}{\nu}\sum_{k=0}^{\infty}\frac{r_{2k+2}(2i\nu)}{(2k+1)!}B_{k+\frac{1}{2}}(-z).\end{aligned}$$

The remainder of the proof is focussed on evaluating the infinite integral. To that end, we make the change of variable $s=\cosh t$ so that $dt=\frac{ds}{\sqrt{s^2-1}}$ and obtain

$$\int_0^\infty e^{-z\cosh t}e^{-\nu t}dt=\int_1^\infty\frac{e^{-zs}e^{-\nu\operatorname{arccosh}s}}{\sqrt{s^2-1}}ds=\frac{e^{-z}}{\nu}-\frac{z}{\nu}\int_1^\infty e^{-zs}e^{-\nu\operatorname{arccosh}s}ds.$$

Thus for fixed $N\in\mathbb{N}$ we may write

$$\int_1^\infty e^{-zs}e^{-\nu\operatorname{arccosh}s}ds=\sum_{k=1}^{N}G_k(z,\nu)+G_\infty(z,\nu),$$

where

$$G_k(z,\nu):=\begin{cases}\displaystyle\int_1^{3/2}e^{-zs}e^{-\nu\operatorname{arccosh}s}ds, & \text{for } k=1,\\[2ex] \displaystyle e^{-kz}\int_{-1/2}^{1/2}e^{zs}e^{-\nu\operatorname{arccosh}(k-s)}ds, & \text{for } k\geq 2,\end{cases}$$

and

$$G_\infty(z,\nu):=\int_{N+1/2}^\infty e^{-zs}e^{-\nu\operatorname{arccosh}s}ds.$$

For each $2\leq k\leq N$, we expand $e^{-\nu\operatorname{arccosh}(k-s)}$ as a power series about $s=0$, with radius of convergence $k-1$. Set

$$h_k(s):=e^{-\nu\operatorname{arccosh}(k-s)}=\sum_{n=0}^{\infty}b_n(k,\nu)s^n.$$

Then we easily find that

$$(s^2-2ks+k^2-1)h_k''(s)=\nu^2h_k(s)+(k-s)h_k'(s).$$

Equating coefficients, we get

$$n(n-1)b_n-2kn(n+1)b_{n+1}+(k^2-1)(n+2)(n+1)b_{n+2}=\nu^2b_n+k(n+1)b_{n+1}-nb_n,$$

and upon rearrangement we have

$$b_{n+2}=\frac{(\nu^2-n^2)b_n+k(n+1)(2n+1)b_{n+1}}{(k^2-1)(n+2)(n+1)},$$

valid for $n \geq 2$ with initial conditions $b_0 = (k + \sqrt{k^2 - 1})^{-\nu}$ and $b_1 = \nu b_0 / \sqrt{k^2 - 1}$. Thus we easily deduce that, for $k \geq 2$,

$$G_k(z, \nu) = \sum_{n=0}^{\infty} e^{-kz} b_n(k, \nu) \beta_n(-z). \tag{6.1}$$

For $G_\infty(z, \nu)$, note that

$$\begin{aligned} s^\nu e^{-\nu \operatorname{arccosh} s} &= s^\nu \left(s + \sqrt{s^2 - 1}\right)^{-\nu} = \left(1 + \sqrt{1 - s^{-2}}\right)^{-\nu} \\ &= \left(1 + \sqrt{1 + (is)^{-2}}\right)^{-\nu} \\ &= (is)^\nu e^{-\nu \operatorname{arcsinh} is}. \end{aligned} \tag{6.2}$$

So by (2.16) we have

$$s^\nu e^{-\nu \operatorname{arccosh} s} = \sum_{n=0}^{\infty} \frac{(-1)^n A_n(\nu)}{s^{2n}}, \tag{6.3}$$

where $A_n(\nu)$ are the same as in Lemma 2.3. Putting this into the expression for G_∞ and interchanging the order of summation and integration, we get that

$$G_\infty(z, \nu) = \sum_{n=0}^{\infty} (-1)^n A_n(\nu) I_n(N + \tfrac{1}{2}, z, \nu). \tag{6.4}$$

The evaluation of $G_1(z, \nu)$ requires more care, as $e^{-\nu \operatorname{arccosh} s}$ does not have a Taylor expansion about $s = 1$ in powers of $s - 1$. However, it does have an expansion in powers of $u := \sqrt{s - 1}$, valid for $|u| < \sqrt{2}$. This is because

$$\begin{aligned} e^{-\nu \operatorname{arccosh} s} &= \left((s^2 - 1)^{1/2} + s\right)^{-\nu} \\ &= \left((s - 1)^{1/2}(s + 1)^{1/2} + s\right)^{-\nu} \\ &= \left(u(u^2 + 2)^{1/2} + u^2 + 1\right)^{-\nu} \end{aligned}$$

is analytic and single-valued on $|u| < \sqrt{2}$, as $u\sqrt{u^2 + 2} + u^2 + 1$ is never zero. Now we let

$$h(s) := e^{-\nu \operatorname{arccosh} s} = e^{-\nu \operatorname{arccosh}(u^2 + 1)} =: H(u),$$

and expand

$$H(u) = \sum_{n=0}^{\infty} d_n(\nu) u^n.$$

Since $h(s) = h_0(-s)$, we have that

$$(s^2 - 1)h''(s) = \nu^2 h(s) - sh'(s).$$

Then we have

$$\frac{dH}{du} = \frac{dh}{ds}\frac{ds}{du} = 2u\frac{dh}{ds},$$

and

$$\begin{aligned}\frac{d^2H}{du^2} &= \frac{d^2h}{ds^2}\left(\frac{ds}{du}\right)^2 + \frac{dh}{ds}\frac{d^2s}{du^2}\\ &= \frac{1}{s^2-1}\left(\nu^2 h - s\frac{dh}{ds}\right)\left(\frac{ds}{du}\right)^2 + \frac{dh}{ds}\frac{d^2s}{du^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}u^2(u^2+2)\frac{d^2H}{du^2} &= \left(\nu^2 H - \left(\frac{u^2+1}{2u}\right)\frac{dH}{du}\right)(4u^2) + \frac{u^2(u^2+2)}{2u}\frac{dH}{du}\times 2\\ &= 4u^2\nu^2 H - u^3\frac{dH}{du}.\end{aligned}$$

Equating coefficients of u^n, we find that

$$n(n-1)d_n + 2(n+2)(n+1)d_{n+2} = 4\nu^2 d_n - nd_n.$$

That is, for $n \geq 0$,

$$d_{n+2} = \frac{4\nu^2 - n^2}{2(n+2)(n+1)}d_n, \tag{6.5}$$

with $d_0 = 1$ and $d_1 = -\nu\sqrt{2}$. Comparing (6.5) with (2.11), we see that $d_n = 2^{-n/2}a_n(0, 2\nu)$. Inserting this back into the expression for G_1, we obtain

$$\begin{aligned}G_1(z,\nu) &= \int_1^{3/2} e^{-zs}e^{-\nu\,\mathrm{arccosh}\,s}ds = \int_0^{1/\sqrt{2}} e^{-z(u^2+1)}H(u)2udu\\ &= 2e^{-z}\sum_{n=0}^{\infty} d_n(\nu)\int_0^{1/\sqrt{2}} e^{-zu^2}u^{n+1}du\\ &= 2e^{-z}\sum_{n=0}^{\infty} 2^{-n/2}a_n(0,2\nu)B_{(n+1)/2}(z/2),\end{aligned} \tag{6.6}$$

where $B_k(p)$ is defined by (2.6). Combining Theorem 2.1, (3.2), (6.1), (6.4), and (6.6) yields the series for I_ν. Similarly, combining (3.3), (6.1), (6.4), and (6.6) yields the series for K_ν.

Finally, we deal with the case $\nu = 0$. The representation for $I_0(z)$ is obvious. For K_0, we have by (3.3),

$$\begin{aligned}K_0(z) &= \int_0^\infty e^{-z\cosh t}dt = \int_1^\infty \frac{e^{-zs}}{\sqrt{s^2-1}}\,ds\\ &= \int_1^{3/2} \frac{e^{-zs}}{\sqrt{s-1}\sqrt{s+1}}\,ds + \sum_{k=2}^{N} e^{-kz}\int_{k-1/2}^{k+1/2}\frac{e^{zs}}{\sqrt{(k-s)^2-1}}\,ds\\ &\quad + \int_{N+1/2}^{\infty}\frac{e^{-zs}}{\sqrt{s^2-1}}\,ds.\end{aligned}$$

If we let

$$\frac{1}{\sqrt{(k-s)^2-1}} = \sum_{n=0}^{\infty} b_n^*(k)s^n,$$

then since

$$\frac{d}{ds}\frac{1}{\sqrt{(k-s)^2-1}} = \frac{k-s}{(k-s)^2-1}\frac{1}{\sqrt{(k-s)^2-1}},$$

we readily find that

$$(k^2-1)(n+1)b^*_{n+1} - 2knb^*_n + (n-1)b^*_{n-1} = kb^*_n - b^*_{n-1},$$

and thus

$$b^*_{n+1} = \frac{k(2n+1)b^*_n - nb^*_{n-1}}{(k^2-1)(n+1)},$$

with $b^*_0 = (k^2-1)^{-1/2}$ and $b^*_1 = kb^*_0/(k^2-1)$. Note also that

$$\frac{1}{\sqrt{s^2-1}} = \frac{1}{s\sqrt{1+(is)^{-2}}} = \sum_{n=0}^{\infty} \frac{(-1)^n a^*_{2n}(0)}{s^{2n+1}}.$$

Therefore, for each $N \in \mathbb{N}$,

$$\int_{3/2}^{\infty} \frac{e^{-zs}}{\sqrt{s^2-1}}\, ds = \sum_{n=0}^{\infty} \left(\beta_n(-z) \sum_{k=2}^{N} e^{-kz} b^*_n(k) + (-1)^n a^*_{2n}(0) I_n(N+\tfrac{1}{2}, z, 1) \right).$$

So, to prove (3.13), it remains to show that

$$\int_{1}^{3/2} \frac{e^{-zs}}{\sqrt{s-1}\sqrt{s+1}}\, ds = \sqrt{2}e^{-z} \sum_{n=0}^{\infty} d^*_n B_n(z/2).$$

To that end, we make a substitution $u = \sqrt{s-1}$ and expand $(u^2+2)^{-1/2}$ as a series in u about $u=0$. That is,

$$\frac{1}{\sqrt{u^2+2}} = \frac{1}{\sqrt{2}}\left(1+\frac{u^2}{2}\right)^{-1/2} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{u^{2n}}{2^n},$$

valid for $|u| < \sqrt{2}$. Thus,

$$\begin{aligned}
\int_{1}^{3/2} \frac{e^{-zs}}{\sqrt{s-1}\sqrt{s+1}}\, ds &= e^{-z} \int_{0}^{1/\sqrt{2}} \frac{2e^{-zu^2}}{\sqrt{u^2+2}}\, du \\
&= \sqrt{2}e^{-z} \sum_{n=0}^{\infty} 2^{-n} \binom{-1/2}{n} \int_{0}^{1/\sqrt{2}} e^{-zu^2} u^{2n}\, du \\
&= \sqrt{2}e^{-z} \sum_{n=0}^{\infty} d^*_n B_n(z/2),
\end{aligned}$$

completing the proof. $\square$

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Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7 Canada
E-mail address: `dborwein@uwo.ca`

Faculty of Computer Science, Dalhousie University, Halifax, Nova Scotia, B3H 1W5, Canada
E-mail address: `jborwein@cs.dal.ca`

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 3J5 Canada
E-mail address: `math@oyeat.com`