II Computer-assisted Discovery and Proof

Seminar
Australian National University
(November 14, 2008)

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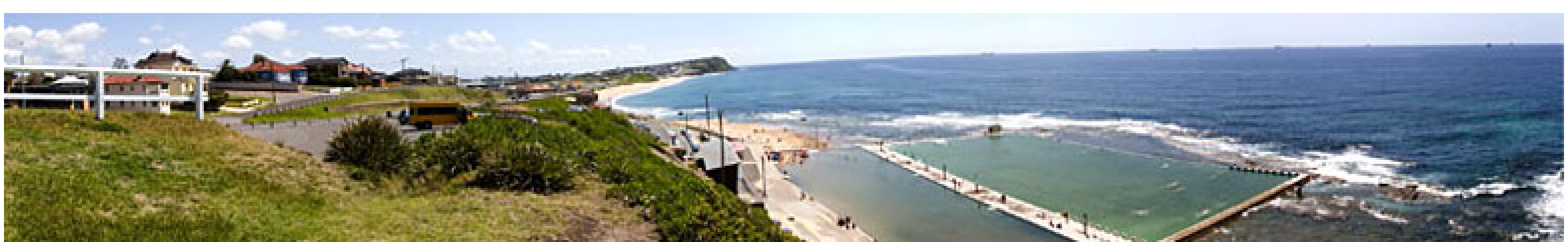
``I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate .... The spoken word and the written word are quite different arts .... I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car."

Sir Lawrence Bragg



Computer-assisted Discovery and Proof
(esp. of Generating Functions for Riemann's Zeta)

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Dalhousie and Newcastle

David H Bailey
Lawrence Berkeley Labs

ABSTRACT I shall further describe some of the methods that are now available for computer-assisted discovery and proof of highly non-trivial mathematical formulas and identities.

Details are in **Excursions in Experimental Mathematics,** Bailey, Borwein et al, A.K. Peters, 2007. (MAA Short Course)

"*Elsewhere Kronecker said* ***"In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas."*** *... I would rather say "computations" than "formulas", but my view is essentially the same.* Harold M. Edwards, 2005

OUTLINE

Part A Underlying Methods and Some Discoveries

- **Individual Digits of Pi**
  - and Normality
- **Reciprocal Series for Pi**
  - Discovery and Replication
  - Proof via Wilf-Zeilberger

Part B Quadrature and Generating Functions

- **High Precision Numerical Quadrature**
  - Theory and Illustrations
- **Apéry like series for $\zeta(2n)$**
  - Discovery and Proof

Conclusions First

- The students of **2010** live in an information-rich, judgement-poor world
- The explosion of information is not going to diminish
- So we have to teach judgement (not obsessive concern with plagiarism)
  - that means mastering the sorts of tools I have illustrated
- We also have to acknowledge that most of our classes will contain a very broad variety of skills and interests (few future mathematicians)
  - properly balanced, discovery and proof can live side-by-side and allow for the mediocre and the talented to flourish in their own fashion

- **Impediments** to the assimilation of the tools I have illustrated are myriad (as I am only too aware from recent teaching experiences)
- These impediments include our own inertia and
  - organizational and technical bottlenecks (IT - not so much dollars)
  - under-prepared or mis-prepared colleagues
  - the dearth of good material from which to teach a modern syllabus

"The plural of 'anecdote' is not 'evidence'."
- Alan L. Leshner, *Science*'s publisher

Algorithms Used in Experimental Mathematics

- Symbolic computation for algebraic and calculus manipulations.
- Integer-relation methods, especially the “PSLQ” algorithm.
- High-precision integer and floating-point arithmetic.
- High-precision evaluation of integrals and infinite series summations.
- The Wilf-Zeilberger algorithm for proving summation identities.
- Iterative approximations to continuous functions.
- Identification of functions based on graph characteristics.
- Graphics and visualization methods targeted to mathematical objects.

"High-Precision" or "Arbitrary Precision" Arithmetic

- High-precision integer arithmetic is required in symbolic computing packages.
- High-precision floating-point arithmetic is required to permit identification of mathematical constants using PSLQ or online constant recognition facilities.
- Most common requirement is for 200-500 digits, although more than 1,000-digit precision is sometimes required.
- One problem required 50,000-digit arithmetic.

"Rigour is the affair of philosophy, not of mathematics."
Bonaventura Cavalieri (1598 -1647)

"Logic is the hygiene the mathematician practices to keep his ideas healthy and strong."
Hermann Weyl, 1885 - 1955

Typical Scheme for High-Precision Floating-Point Arithmetic

iDrive

A high-precision number is represented as a string of n + 4 integers (or a string of n + 4 floating-point numbers with integer values):

- First word contains sign and n, the number of words.
- Second word contains p, the exponent (power of 2^b).
- Words three through n + 2 contain mantissas m_1 through m_n.
- Words n + 3 and n + 4 are for convenience in arithmetic.
- The value is then given by:

$$A = \pm(2^{pb}m_1 + 2^{(p-1)b}m_2 + 2^{(p-2)b}m_3 + \cdots + 2^{(p-n+1)b}m_n)$$

For **basic arithmetic operations**, conventional schemes suffice up to about 1000 digits. Beyond that level, Karatsuba's algorithm or FFTs, or… can be used for significantly faster multiply performance.

Division and square roots can be performed by Newton iterations.

For **transcendental functions**, Taylor series or (for higher precision) quadratically convergent elliptic function algorithms can be used (Brent)

Newton Iteration Methods

Newton iterations arise frequently in experimental math, such as to iteratively solve an equation p(x) = 0:

$$x_{k+1} = x_k - \frac{p(x)}{p'(x)}$$

Applications include:

- Performing division and square roots using high-precision arithmetic.
- Computing exp, given a fast scheme for log.
- Finding polynomial roots and roots of more general functions.

Potential pitfalls:

- A large array of values may need to be computed to locate the root.
- Derivative of function may be zero at a zero of the function.

See companion book for ways to deal with such problems.

LBNL's Arbitrary Precision Computation (ARPREC) Package

- Low-level routines written in C++.
- C++ and F-90 translation modules permit use with existing programs with only minor code changes.
- Double-double (32 digits), quad-double, (64 digits) and arbitrary precision (>64 digits) available.
- Special routines for extra-high precision (>1000 dig).
  - Arithmetic being upgraded (including Brent-Zimmermann)
- Includes common math functions: sqrt, cos, exp, etc.
- PSLQ, root finding, numerical integration.
- An interactive "Experimental Mathematician's Toolkit" employing this software is also available.

Available at: **http://www.experimentalmath.info**

Also recommended: GMP/MPFR package, available at

http://www.mpfr.org

The PSLQ Integer Relation Algorithm

Let (x_n) be a vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

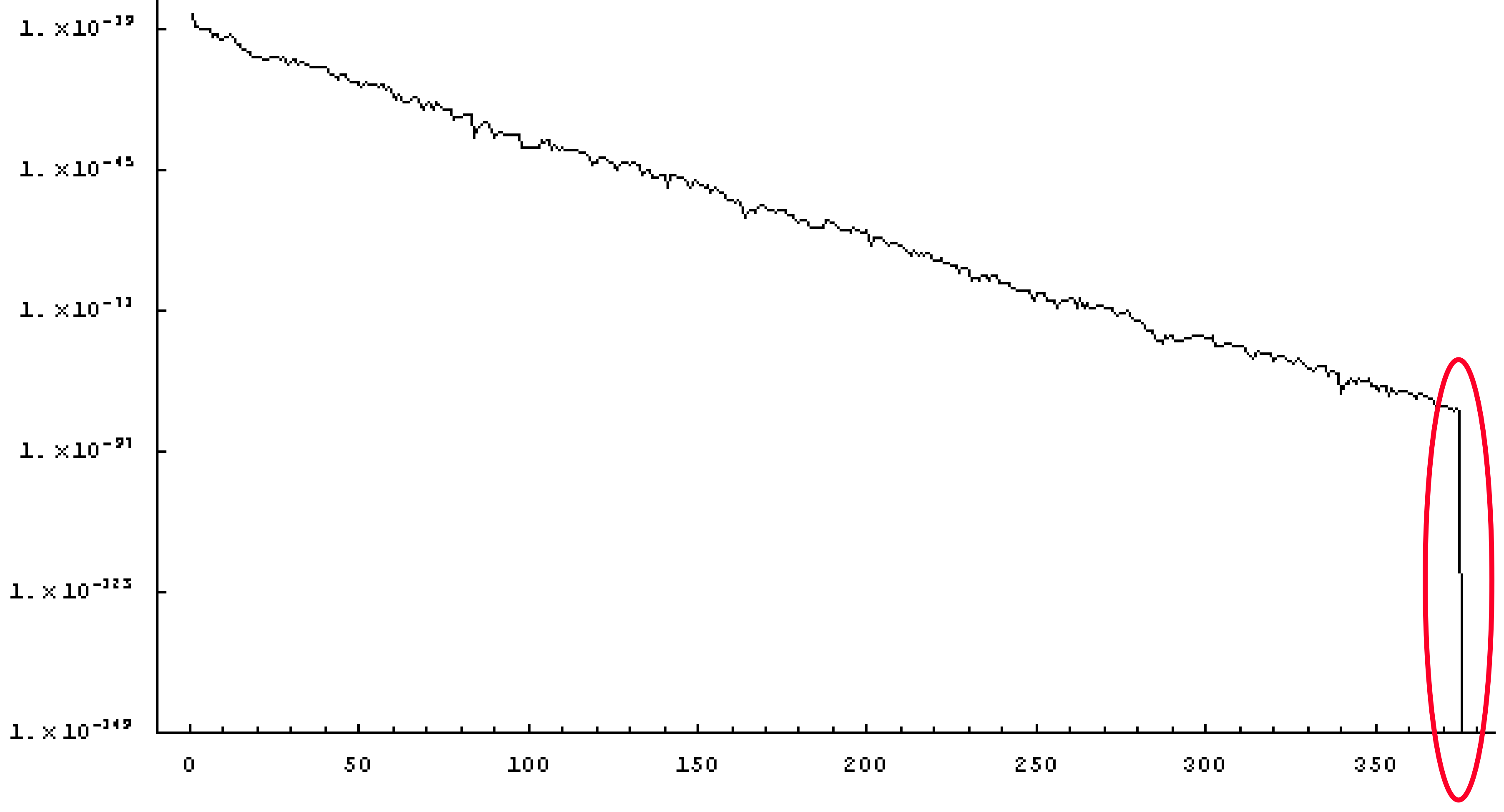
- At the present time, the PSLQ algorithm of mathematician-sculptor *Helaman Ferguson* is the best-known integer relation algorithm.
- PSLQ was named one of ten "algorithms of the century" by *Computing in Science and Engineering*.
- High precision arithmetic software is required: at least $d \times n$ digits, where d is the size (in digits) of the largest of the integers a_k.

Peter Borwein in front of Helaman Ferguson's work

CMS Meeting
December 2003
SFU Harbour Centre

Ferguson uses high tech tools and micro engineering at NIST to build monumental math sculptures

Decrease of $\min_j |A_j x|$ in PSLQ: self-diagnosing



Some Supercomputer-Class PSLQ Solutions

- **Identification of B_4**, the fourth bifurcation point of the logistic iteration.
  - Integer relation of size 121; 10,000 digit arithmetic.
- **Identification of Apéry sums**.
  - 15 integer relation problems, with dimension up to 118, requiring up to 5,000 digit arithmetic.
- **Identification of Euler-zeta sums**.
  - Hundreds of integer relation problems, each of size 145 and requiring 5,000 digit arithmetic.
  - Run on IBM SP parallel system.
- **Finding relation for root of Lehmer's polynomial**.
  - Integer relation of size 125; 50,000 digit arithmetic.
  - Utilizes 3-level, multi-pair parallel PSLQ program.
  - Run on IBM SP using ARPEC; 16 hours on 64 CPUs.

Fascination With Pi

Newton (1670):

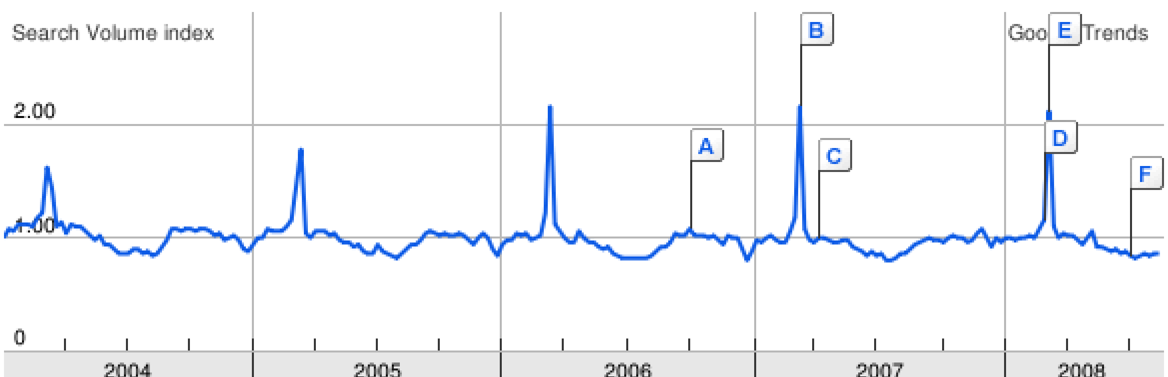
"I am ashamed to tell you to how many figures I carried these computations, having no other business at the time." (1666)

Carl Sagan (1986):

In his book "Contact," the lead scientist (played by Jodie Foster in the movie) looked for patterns in the digits of pi.

Wall Street Journal (Mar 15, 2005):

"Yesterday was Pi Day: March 14, the third month, 14th day. As in 3.14, roughly the ratio of a circle's circumference to its diameter…"



GoogleFlu

NYT 3.15.08

Fax from "The Simpsons" Show

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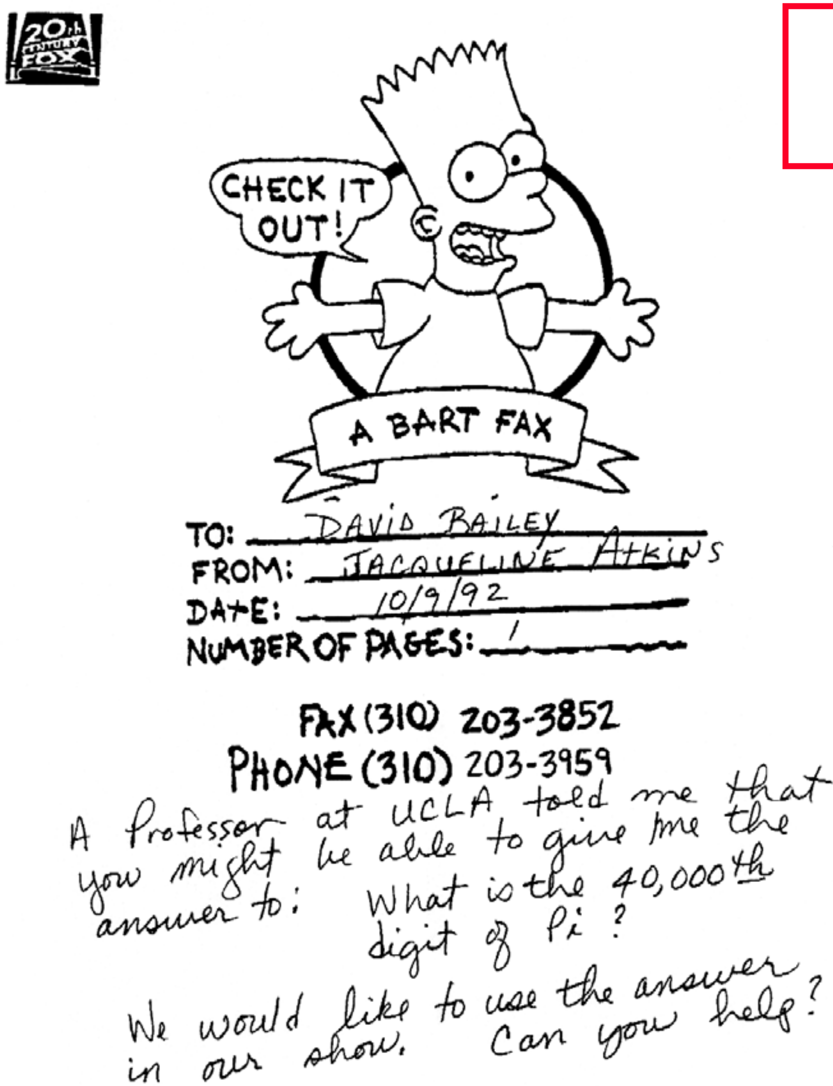

CHECK IT OUT!

A BART FAX

TO: DAVID BAILEY

FROM: JACQUELINE ATKINS

DATE: 10/9/92

NUMBER OF PAGES: 1

FAX (310) 203-3852

PHONE (310) 203-3959

A Professor at UCLA told me that you might be able to give me the answer to: What is the 40,000th digit of Pi?

We would like to use the answer in our show. Can you help?

Peter Borwein's Observation

In 1996, Peter Borwein of SFU in Canada observed that the following well-known formula for $\log_e 2$

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n 2^n} = 0.693147180559945309 42\ldots$$

leads to a simple scheme for computing binary digits at an arbitrary starting position (here {} denotes fractional part):

$$\begin{aligned}\left\{2^d \log 2\right\} &= \left\{\sum_{n=1}^{d} \frac{2^{d-n}}{n}\right\} + \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \\ &= \left\{\sum_{n=1}^{d} \frac{2^{d-n} \bmod n}{n}\right\} + \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n}\end{aligned}$$

Fast Exponentiation Mod n

The exponentiation (2^{d-n} mod n) in this formula can be evaluated very rapidly by means of the binary algorithm for exponentiation, performed modulo n:

Example:

$3^{17} = (((3^2)^2)^2)^2 \times 3 = 129140163$

In a similar way, we can evaluate:

$3^{17} \bmod 10 = ((((3^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10) \times 3 \bmod 10$

$3^2 \bmod 10 = 9$

$9^2 \bmod 10 = 1$

$1^2 \bmod 10 = 1$

$1^2 \bmod 10 = 1$

$1 \times 3 = 3$ Thus $3^{17} \bmod 10 = 3$.

Note: we never have to deal with integers larger than 81.

Is There a BBP-Type Formula for Pi?

The “trick” for computing digits beginning at an arbitrary position in the binary expansion of log(2) works for any constant that can be written with a formula of the form

$$\alpha = \sum_{n=1}^{\infty} \frac{p(n)}{2^n q(n)}$$

where p and q are polynomial functions with integer coefficients, and q has no zeroes at positive integer values.

- In 1995, no formula of this type was known for π.

Note however that if α and β have such a formula, then so does $\gamma = r\,\alpha + s\,\beta$, where r and s are integers. This suggests using PSLQ to find a formula for π.

The BBP Formula for Pi

In 1996, Simon Plouffe at PBB's suggestion, using DHB's PSLQ program, discovered this formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

Indeed, this formula permits one to directly calculate binary or hexadecimal (base-16) digits of π beginning at an arbitrary starting position n, without needing to calculate any of the first n-1 digits.

It was found after several months in terms of hypergeometric and logarithmic values that reduced to the above.

Was used to `quickly' confirm Kanada's 2002 computation of a trillion hex digits (and 1.25 trillion decimals).

Proof of the BBP Formula (Maple or Mathematica)

$$\int_0^{1/\sqrt{2}} \frac{x^{j-1}\,dx}{1-x^8} = \int_0^{1/\sqrt{2}} \sum_{k=0}^{\infty} x^{8k+j-1}\,dx = \frac{1}{2^{j/2}} \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}$$

Thus

$$\sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

$$= \int_0^{1/\sqrt{2}} \frac{(4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5)\,dx}{1-x^8}$$

$$= \int_0^1 \frac{16(4 - 2y^3 - y^4 - y^5)\,dy}{16 - y^8}$$

$$= \int_0^1 \frac{16(y-1)\,dy}{(y^2-2)(y^2-2y+2)}$$

$$= \int_0^1 \frac{4y\,dy}{y^2-2)} - \int_0^1 \frac{(4y-8)\,dy}{y^2-2y+2}$$

$$= \pi$$

Calculations Using the BBP Algorithm

Position	Hex Digits of Pi Starting at Position	
10^6	26C65E52CB4593	
10^7	17AF5863EFED8D	
10^8	ECB840E21926EC	
10^9	85895585A0428B	
10^{10}	921C73C6838FB2	
10^{11}	9C381872D27596	
1.25×10^{12}	07E45733CC790B	[1]
2.5×10^{14}	E6216B069CB6C1	[2]

[1] Fabrice Bellard, France, 1999

[2] Colin Percival, Canada, 2000

Some Other Similar New Identities

$$\pi\sqrt{3} = \frac{9}{32}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{16}{6k+1}-\frac{8}{6k+2}-\frac{2}{6k+4}-\frac{1}{6k+5}\right)$$

$$\pi^2 = \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{144}{(6k+1)^2}-\frac{216}{(6k+2)^2}-\frac{72}{(6k+3)^2}-\frac{54}{(6k+4)^2}+\frac{9}{(6k+5)^2}\right)$$

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left(\frac{243}{(12k+1)^2}-\frac{405}{(12k+2)^2}-\frac{81}{(12k+4)^2}-\frac{27}{(12k+5)^2}\right.$$
$$\left.-\frac{72}{(12k+6)^2}-\frac{9}{(12k+7)^2}-\frac{9}{(12k+8)^2}-\frac{5}{(12k+10)^2}+\frac{1}{(12k+11)^2}\right)$$

$$6\sqrt{3}\arctan\left(\frac{\sqrt{3}}{7}\right) = \sum_{k=0}^{\infty}\frac{1}{27^k}\left(\frac{3}{3k+1}+\frac{1}{3k+2}\right)$$

$$\frac{25}{2}\log\left(\frac{781}{256}\left(\frac{57-5\sqrt{5}}{57+5\sqrt{5}}\right)^{\sqrt{5}}\right) = \sum_{k=0}^{\infty}\frac{1}{5^{5k}}\left(\frac{5}{5k+2}+\frac{1}{5k+3}\right)$$

Is There a Base-10 Formula for Pi?

Note that there is both a base-2 and a base-3 BBP-type formula for π^2. Base-2 and base-3 formulas are also known for a handful of other constants.

Question: Is there any base-n BBP-type formula for π, where n is NOT a power of 2?

Answer: No. This is ruled out in a 2004 paper by JMB, David Borwein and Will Galway.

This does not rule out some completely different scheme for finding individual non-binary digits of π.

Q1 Is there any **natural** slow series for e?

Normal Numbers

- A number is b-normal (or "normal base b") if every string of m digits in the base-b expansion appears with limiting frequency b^{-m}.
- Using measure theory, it is easy to show that almost all real numbers are b-normal, for any b.
- Widely believed to be b-normal, for any b:
  - π = 31415.926535…
  - e = 2.7182818284…
  - Sqrt(2) = 1.4142135623…
  - Log(2) = 0.6931471805…
  - All irrational roots of polynomials with integer coefficients.

But to date there have been no proofs for any of these.
Proofs have been known only for contrived examples, such as C = 0.12345678910111213…

BBP Formulas and Normality

Consider the 'chaotic' sequence defined by $x_0 = 0$, and

$$x_n = \left\{2x_{n-1} + \frac{1}{n}\right\}$$

where { } denotes fractional part as before.

Result: log(2) is 2-normal if and only if this sequence is equidistributed in the unit interval.

In a similar vein, consider the sequence $x_0 = 0$, and

$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\}$$

Result: π is 16-normal (and hence 2-normal) if and only if this sequence is equidistributed in the unit interval: $\lfloor 16x_n \rfloor$ agrees with the hex-digits of Pi for a million terms and probably forever (Borel law).

A Class of Provably Normal Constants

Crandall and DHB have also shown (unconditionally) that an infinite class of mathematical constants is normal, including

$$\begin{aligned}\alpha_{2,3} &= \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}} \\ &= 0.041883680831502985071252898624571682426096\ldots_{10} \\ &= \mathtt{0.0AB8E38F684BDA12F684BF35BA781948B0FCD6E9E0}\ldots_{16}\end{aligned}$$

$\alpha_{2,3}$ was proven 2-normal by Stoneham in 1971, but we have extended this to the case where (2,3) are any pair (p,q) of relatively prime integers. We also extended to uncountably infinite class, as follows [here r_k is the k-th bit of r in (0,1)]:

$$\alpha_{2,3}(r) = \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k + r_k}}$$

Ramanujan-Like Identities Revisited

Guillera and Gourevich have recently found Ramanujan-like identities, including:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n}$$
$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n}$$
$$\frac{32}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n}.$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$$

Guillera proved the first two of these using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods.

Are there any higher-order analogues?

Not as far as we can tell

Searches for Additional Formulas

We searched for additional formulas of either the following forms:

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}$$

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} (-1)^n r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}.$$

where c is some linear combination of

$1,\ 2^{1/2},\ 2^{1/3},\ 2^{1/4},\ 2^{1/6},\ 4^{1/3},\ 8^{1/4},\ 32^{1/6},\ 3^{1/2},\ 3^{1/3},\ 3^{1/4},\ 3^{1/6},\ 9^{1/3},$
$27^{1/4},\ 243^{1/6},\ 5^{1/2},\ 5^{1/4},\ 125^{1/4},\ 7^{1/2},\ 13^{1/2},\ 6^{1/2},\ 6^{1/3},\ 6^{1/4},\ 6^{1/6},$
$7, 36^{1/3},\ 216^{1/4},\ 7776^{1/6},\ 12^{1/4},\ 108^{1/4},\ 10^{1/2},\ 10^{1/4},\ 15^{1/2}$

where each of the coefficients p_i is a linear combination of

$1,\ 2^{1/2},\ 3^{1/2},\ 5^{1/2},\ 6^{1/2},\ 7^{1/2},\ 10^{1/2},\ 13^{1/2},\ 14^{1/2},\ 15^{1/2},\ 30^{1/2}$

and where α is chosen as one of the following:

$1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32,\ 1/64,\ 1/128,\ 1/256,\ \sqrt{5}-2,\ (2-\sqrt{3})^2,$
$5\sqrt{13}-18,\ (\sqrt{5}-1)^4/128,\ (\sqrt{5}-2)^4,\ (2^{1/3}-1)^4/2,\ 1/(2\sqrt{2}),$
$(\sqrt{2}-1)^2,\ (\sqrt{5}-2)^2,\ (\sqrt{3}-\sqrt{2})^4$

Relations Found by PSLQ

in addition to Guillera's three relations were all known via Ramanujan-like methods and none were missed

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} r(n)^3(1+6n)\left(\frac{1}{2}\right)^{2n}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} r(n)^3(5+42n)\left(\frac{1}{8}\right)^{2n}$$

$$\frac{12^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-15+9\sqrt{3}-36n+24\sqrt{3}n)\left(2-\sqrt{3}\right)^{4n}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-1+5\sqrt{5}+30n+42\sqrt{5}n)\left(\frac{(\sqrt{5}-1)^4}{128}\right)^{2n}$$

$$\frac{5^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-525+235\sqrt{5}-1200n+540\sqrt{5}n)\left(\sqrt{5}-2\right)^{8n}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(1+6n)\left(\frac{1}{2\sqrt{2}}\right)^{2n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(-5+4\sqrt{2}-12n+12\sqrt{2}n)\left(\sqrt{2}-1\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(23-10\sqrt{5}+60n-24\sqrt{5}n)\left(\sqrt{5}-2\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(177-72\sqrt{6}+420n-168\sqrt{6}n)\left(\sqrt{3}-\sqrt{2}\right)^{8n}$$

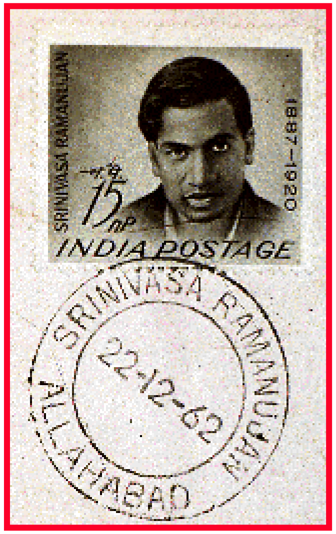
The Wilf-Zeilberger Algorithm for Proving Identities

- A slick, computer-assisted proof scheme to prove certain types of identities
- Provides a nice complement to PSLQ

  - **PSLQ and the like permit one to discover new identities but do not constitute rigorous proof**

  - **W-Z methods permit one to prove certain types of identities but do not suggest any means to discover the identity**

Example Usage of W-Z



We rewrite two experimentally-discovered identities

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4}{2^{16n}} \left(120n^2 + 34n + 3\right) = \frac{32}{\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} \left(820n^2 + 180n + 13\right) = \frac{128}{\pi^2}$$

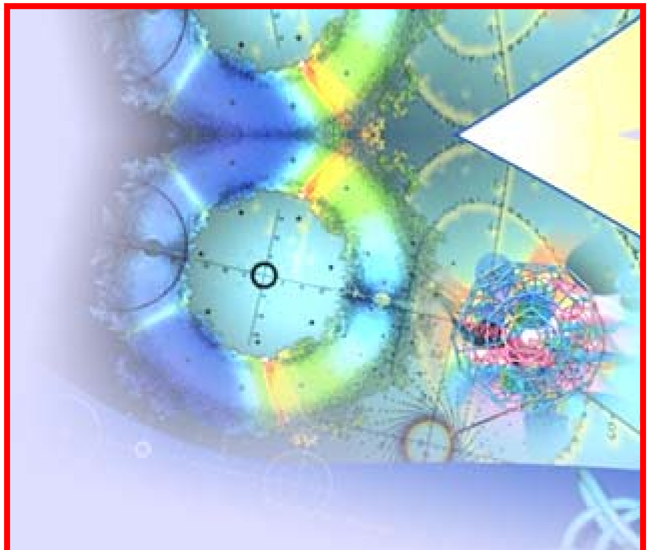
Guillera ***cunningly*** started by defining

$$G(n,k) = \frac{(-1)^k}{2^{16n} 2^{4k}} \left(120n^2 + 84nk + 34n + 10k + 3\right) \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k} \binom{n+k}{n}^2}$$

He then used the **EKHAD** software package to obtain the companion

$$F(n,k) = \frac{(-1)^k 512}{2^{16n} 2^{4k}} \frac{n^3}{4n-2k-1} \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k} \binom{n+k}{n}^2}$$

Example Usage of W-Z, II



http://ddrive.cs.dal.ca/~isc/portal

When we define

$$H(n,k) \;=\; F(n+1,n+k)+G(n,n+k)$$

Zeilberger's theorem gives the identity

$$\sum_{n=0}^{\infty} G(n,0) \;=\; \sum_{n=0}^{\infty} H(n,0)$$

which when written out is

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4 \binom{4n}{2n}}{2^{16n}} \left(120n^2+34n+3\right) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n+7}} \frac{(n+1)^3}{2n+3} \frac{\binom{2n+2}{n+1}^4 \binom{2n}{n}^3 \binom{2n+4}{n+2}}{\binom{2n+2}{n}\binom{2n+1}{n+1}^2}$$

$$+\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n}} \left(204n^2+44n+3\right)\binom{2n}{n}^5 \;=\; \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} \left(820n^2+180n+13\right)$$

A limit argument completes the proof of Guillera’s identities

Q2. What about the formula for $1/\pi^3$?

A Cautionary Example

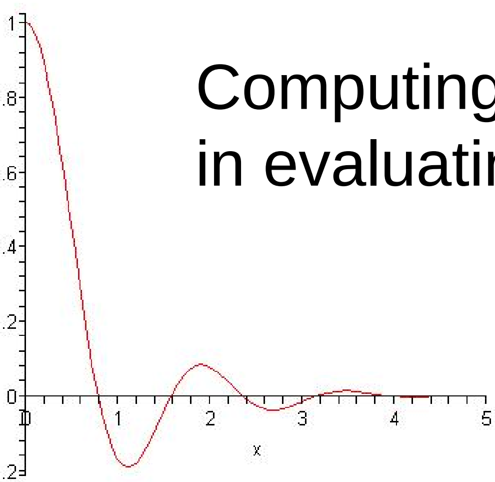
These **constants agree to 42 decimal digits** accuracy, but are **NOT** equal:

$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos(x/n)\, dx \quad =$$

$$0.39269908169872415480783042290993786052464\mathbf{543418723}\ldots$$

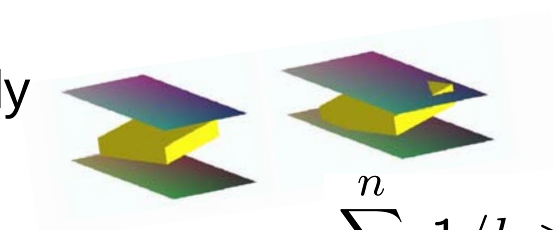
$$\frac{\pi}{8} \quad =$$

$$0.39269908169872415480783042290993786052464\mathbf{617492189}\ldots$$



Computing this integral is nontrivial, due largely to difficulty in evaluating the integrand function to high precision.

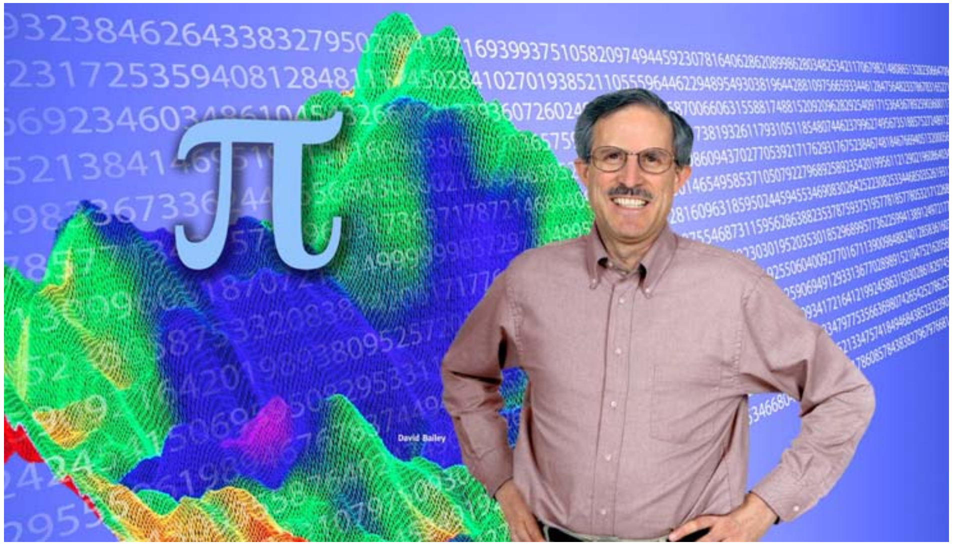
Fourier transforms turn the integrals into volumes and neatly explains this happens when a hyperplane meets a hypercube (LP) …



$$\sum_{k=2}^{n} 1/k > 2$$

Part B

JM Borwein and DH Bailey



“Anyone who is not shocked by quantum theory has not understood a single word." - Niels Bohr

History of Numerical Quadrature

- **1670**: Newton devises Newton-Coates integration.
- **1740**: Thomas Simpson develops Simpson's rule.
- **1820**: Gauss develops Gaussian quadrature.
- **1950-1980**: Adaptive quadrature, Romberg integration, Clenshaw-Curtis integration, others.
- **1985-1990**: Maple and Mathematica feature built-in numerical quadrature facilities.
- **2000**: Very high-precision quadrature (1000+ digits).

With these high-precision values, we can use PSLQ to obtain analytical evaluations of integrals: **Monte Carlo** and **Sparse Grid** methods are not usually adequate. We are currently trying to use a mixture of Sparse Grid and what follows:

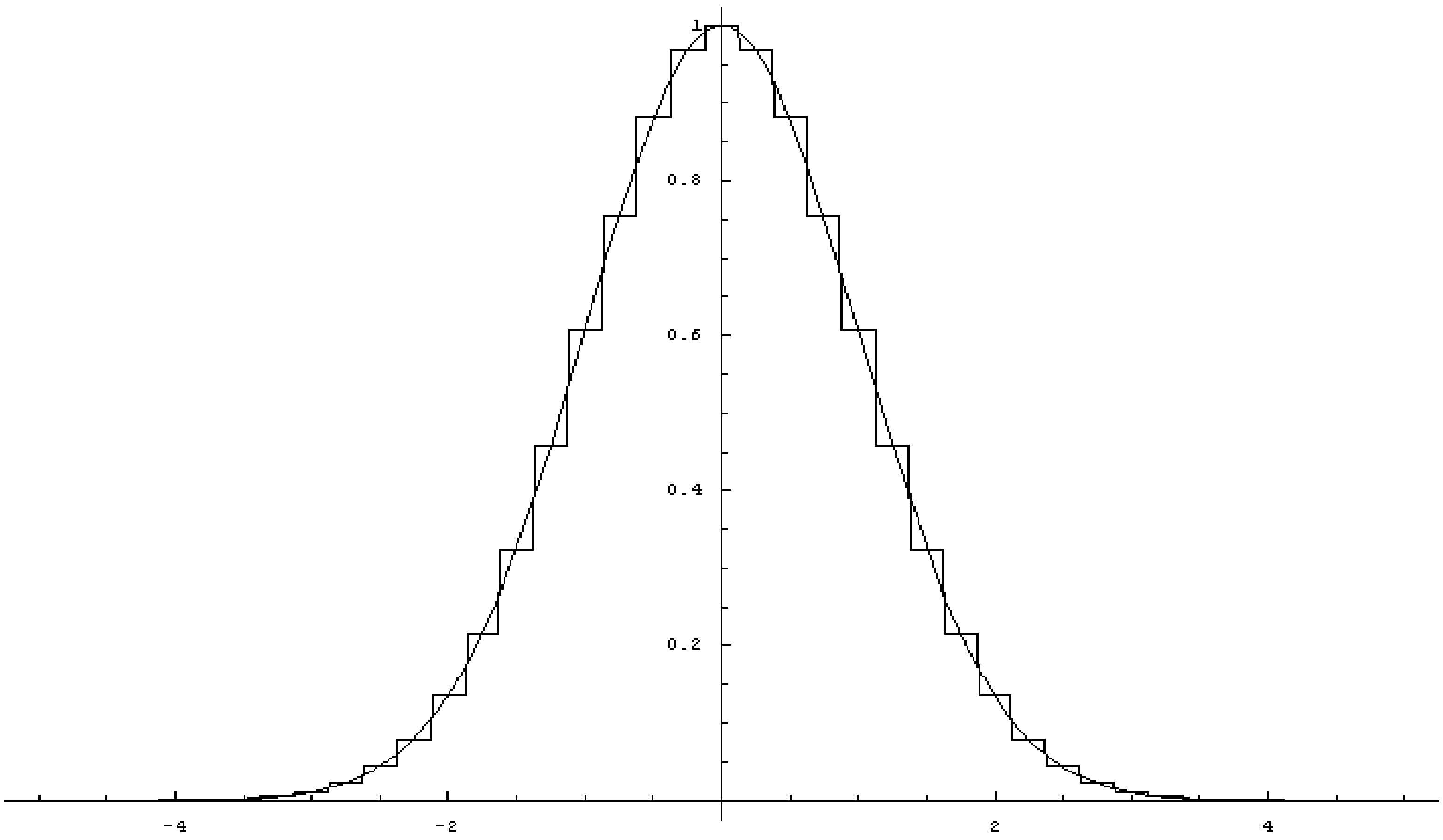
The Euler-Maclaurin Formula

$$\begin{aligned}\int_a^b f(x)\,dx &= h\sum_{j=0}^{n} f(x_j) - \frac{h}{2}(f(a)+f(b)) \\ &- \sum_{i=1}^{m}\frac{h^{2i}B_{2i}}{(2i)!}\Big(D^{2i-1}f(b) - D^{2i-1}f(a)\Big) - E(h) \\ |E(h)| &\le 2(b-a)[h/(2\pi)]^{2m+2}\max_{a\le x\le b}|D^{2m+2}f(x)|\end{aligned}$$

[Here h = (b - a)/n and x_j = a + j h. D^m f(x) means m-th derivative of f(x).]

Note when f(t) and all of its derivatives are zero at a and b, the error E(h) of a simple block-function approximation to the integral goes to zero more rapidly than any power of h.

Block-Function Approximation to the Integral of a Bell-Shaped Function



Quadrature and the Euler-Maclaurin Formula

Given f(x) defined on (-1,1), employ a function g(t) such that g(t) goes from -1 to 1 over the real line, with g'(t) going to zero for large |t|. Then substituting x = g(t) yields

$$\int_{-1}^{1} f(x)\,dx = \int_{-\infty}^{\infty} f(g(t))g'(t)\,dt$$

$$\approx h\sum_{-N}^{N} g'(hj)f(g(hj)) = h\sum_{-N}^{N} w_j f(x_j)$$

[Here x_j = g(hj) and w_j = g'(hj).]

If g'(t) goes to zero rapidly enough for large t, then even if f(x) has an infinite derivative or blow-up singularity at an endpoint, f(g(t)) g'(t) often is a nice bell-shaped function for which the E-M formula applies.

Three Suitable 'g' Functions

$$g(t) = \operatorname{erf}(t) \qquad g'(t) = \frac{2}{\sqrt{\pi}} e^{-t^2}$$

$$g(t) = \tanh t \qquad g'(t) = \frac{1}{\cosh^2 t}$$

$$g(t) = \tanh(\pi/2 \cdot \sinh t) \qquad g'(t) = \frac{\pi/2 \cdot \sinh t}{\cosh^2(\pi/2 \cdot \sinh t)}$$

$$g(t) = \tanh(\sinh t) \qquad g'(t) = \frac{\sinh t}{\cosh^2(\sinh t)}$$

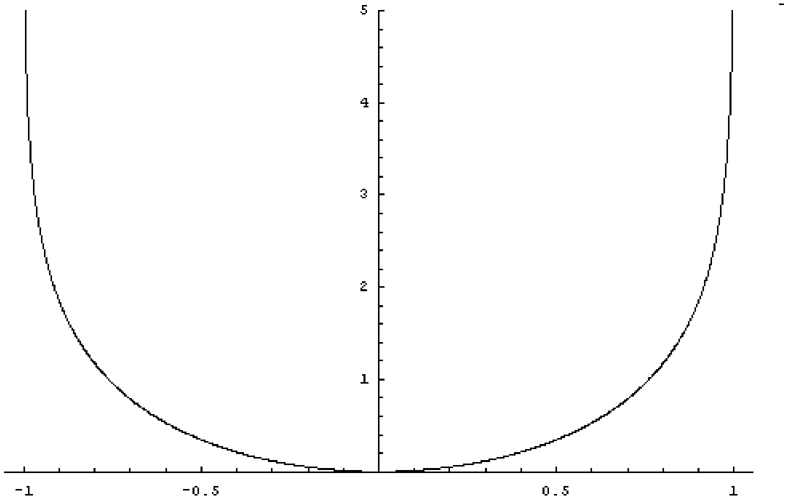
The third & fourth are known as **"tanh-sinh" quadrature**

- are being implemented in *Maple* and *Mathematica*
  - non-adaptive (which is often a virtue)
  - excellent in 1D, good in 2D

Original and Transformed Integrand Function

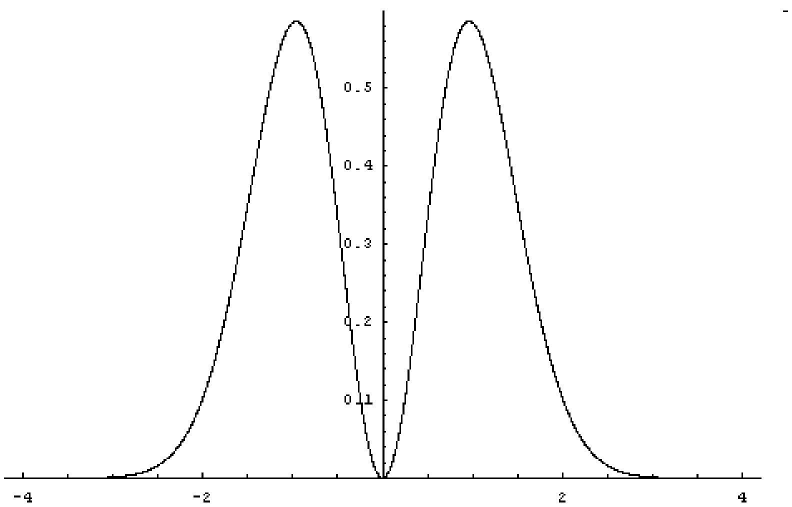
Original function (on [-1,1]):

$$f(t) = -\log\cos\left(\frac{\pi t}{2}\right)$$



Transformed function using g(t) = erf t:

$$f(g(t))g'(t) = -\frac{2}{\sqrt{\pi}}\log\cos\left(\frac{\pi \mathrm{erf}\, t}{2}\right)\exp(-t^2)$$



Tanh-Sinh Quadrature: Example 1

Let

$$C(a) = \int_0^1 \frac{\arctan\sqrt{x^2+a^2}}{(x^2+1)\sqrt{x^2+a^2}} dx$$

Then PSLQ yields

$$\begin{aligned} C(0) &= (\pi \log 2)/8 + G/2 \\ C(1) &= \pi/4 - \pi\sqrt{2}/2 + 3\sqrt{2}/2 \cdot \arctan\sqrt{2} \\ C(\sqrt{2}) &= 5\pi^2/96 \end{aligned}$$

Several general results have now been proven, including

$$\int_0^\infty \frac{\arctan\sqrt{x^2+a^2}\, dx}{(x^2+1)\sqrt{x^2+a^2}} = \frac{\pi}{2\sqrt{a^2-1}}\left(2\arctan\sqrt{a^2-1} - \arctan\sqrt{a^4-1}\right)$$

Numerical Integration and PSLQ

$$\frac{2}{\sqrt{3}}\int_0^1 \frac{\log^6 x \arctan[x\sqrt{3}/(x-2)]}{x+1}\,dx \;=\;$$
$$\frac{1}{81648}\Big(-229635 L_{-3}(8) + 29852550 L_{-3}(7)\log 3$$
$$-1632960 L_{-3}(6)\pi^2 + 27760320 L_{-3}(5)\zeta(3)$$
$$-275184 L_{-3}(4)\pi^4 + 36288000 L_{-3}(3)\zeta(5)$$
$$-30008 L_{-3}3(2)\pi^6 - 57030120 L_{-3}(1)\zeta(7)\Big)$$

where

$$L_{-3}(s) \;=\; \sum_{n=1}^{\infty} \left[1/(3n-2)^s - 1/(3n-1)^s\right]$$

is a primitive Dirichlet series.

Numerical Integration: Example 2

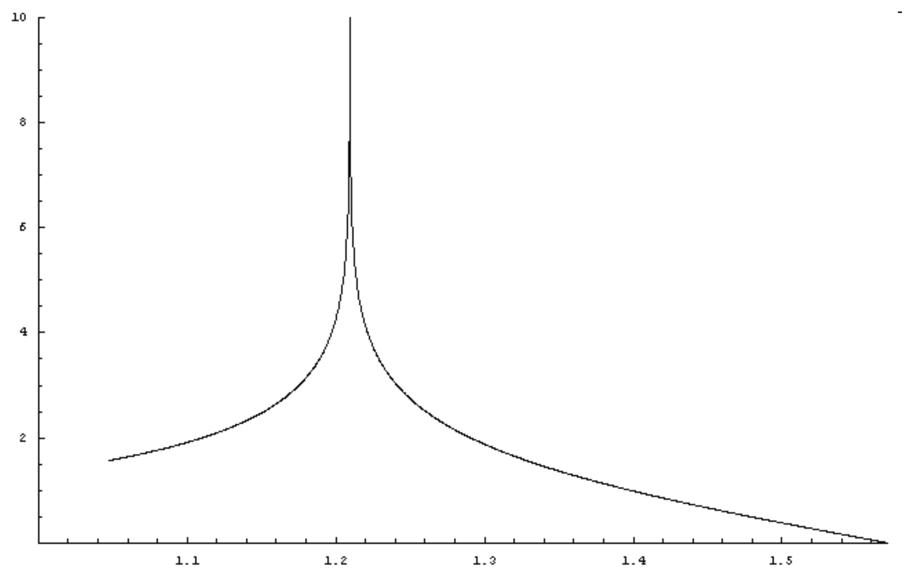
$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right]$$

This arises in mathematical physics, from analysis of the volumes of *ideal tetrahedra* in hyperbolic space.

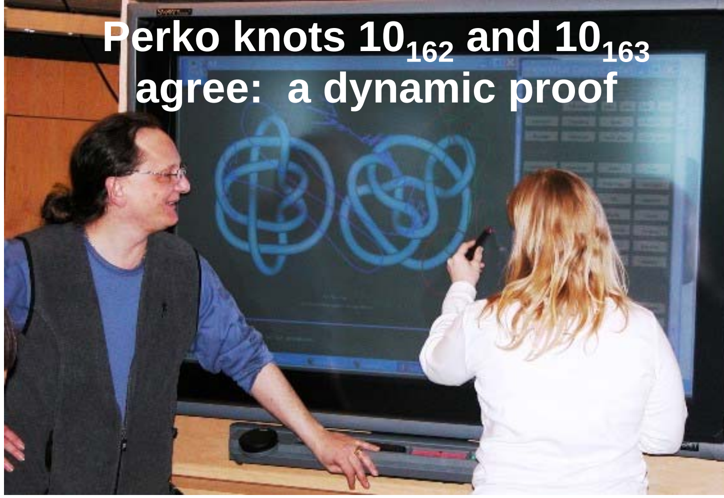
This identity (one of 998) is proven (Zagier 86) and verified numerically to 20,000 digits, but no proof is known of most of the other 997.

Note that the integrand function has a nasty singularity.



Extreme Quadrature …
20,000 Digits (50 Certified) on 1024 CPUs

- The integral was split at the nasty interior singularity
- The sum was `easy'.
- All fast arithmetic & function evaluation ideas used



Run-times and speedup ratios on the Virginia Tech G5 Cluster

CPUs	Init	Integral #1	Integral #2	Total	Speedup
1	*190013	*1534652	*1026692	*2751357	1.00
16	12266	101647	64720	178633	15.40
64	3022	24771	16586	44379	62.00
256	770	6333	4194	11297	243.55
1024	199	1536	1034	2769	993.63

2.1 years

Parallel run times (in seconds) and speedup ratios for the 20,000-digit problem

Expected and unexpected scientific spinoffs

- **1986-1996**. Cray used quartic-Pi to check machines in factory
- **1986**. Complex FFT sped up by factor of two
- **2002**. Kanada used hex-pi (20hrs not 300hrs to check computation)
- **2005**. Virginia Tech (this integral pushed the limits)
- **1995-** Math Resources (another lecture)

Example 3

Define

$$J_n = \int_{n\pi/60}^{(n+1)\pi/60} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt$$

Then

$$\begin{aligned} 0 \stackrel{?}{=} \; & -2J_2 - 2J_3 - 2J_4 + 2J_{10} + 2J_{11} + 3J_{12} \\ & +3J_{13} + J_{14} - J_{15} - J_{16} - J_{17} - J_{18} \\ & -J_{19} + J_{20} + J_{21} - J_{22} - J_{23} + 2J_{25} \end{aligned}$$

This has been verified to over **1000** digits. The interval in J_{23} includes the singularity.

Error Estimation in Tanh-Sinh Quadrature

Let F(t) be the desired integrand function, and then define f(t) = F(g(t)) g'(t), where g(t) = tanh (sinh t) (or one of the other g functions above). Then an estimate of the error of the quadrature result, with interval h, is:

$$E_2(h,m) \;=\; h(-1)^{m-1}\left(\frac{h}{2\pi}\right)^{2m}\sum_{j=a/h}^{b/h} D^{2m} f(jh)$$

- First order (m = 1) estimates are remarkably accurate and "cheap"..
- Higher-order estimates (m > 1) can be used to obtain "certificates" on the accuracy of a tanh-sinh quadrature result.

For convergence analysis see:

JMB and Peter Ye, "Quadratic Convergence of 'tanh-sinh' Quadrature," manuscript, available at

http://users.cs.dal.ca/~jborwein/tanh-sinh.pdf

Example of Error Estimates

Results for tanh-sinh quadrature to integrate the function

$$F(t) \;=\; 1/(1+t^2+t^4+t^6) \quad \text{on} \quad [-1,1]$$

h	$E(h)$	$\lvert E(h)-E_2(h,1)\rvert$	$\lvert E(h)-E_2(h,2)\rvert$
$1/1$	5.34967×10^{-3}	9.81980×10^{-4}	4.77454×10^{-3}
$1/2$	-3.36641×10^{-4}	1.12000×10^{-7}	5.60084×10^{-7}
$1/4$	-3.73280×10^{-8}	1.67517×10^{-16}	8.37583×10^{-16}
$1/8$	5.58389×10^{-17}	2.29357×10^{-32}	1.14679×10^{-31}
$1/16$	-7.64525×10^{-33}	2.07256×10^{-64}	1.03628×10^{-63}
$1/32$	-6.90852×10^{-65}	7.23441×10^{-129}	3.61721×10^{-128}
$1/64$	-2.41147×10^{-129}	9.08805×10^{-259}	4.54403×10^{-258}

DHB and JMB, "Effective Error Estimates in Euler-Maclaurin Based Quadrature Schemes," (published) available at **http://crd.lbl.gov/~dhbailey/dhbpapers/em-error.pdf**

Apéry-Like Summations

The following formulas for $\zeta(n)$ have been known for many decades:

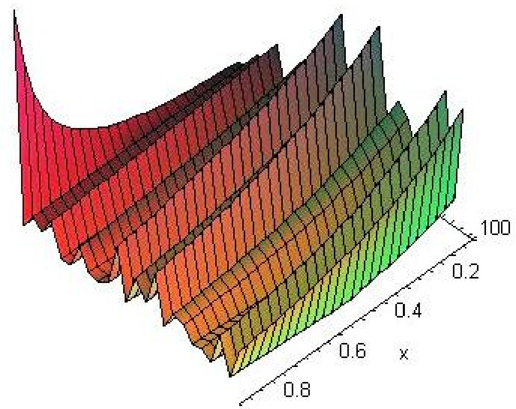
$$\zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}},$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}},$$

Apéry

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}.$$

The RH in Maple



These results have led many to speculate that

$$Q_5 := \zeta(5)/\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}$$

might be some nice rational or algebraic value.

Sadly, PSLQ calculations have established that if Q_5 satisfies a polynomial with **degree** at most **25**, then at least **one coefficient** has **380** digits.

Best result since Apéry (1978) showed $\zeta(3)$ is irrational: provably one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational (Zudilin, Moscow and Newcastle)

Nothing New under the Sun

Margo Kondratieva found a formula of Markov in 1890:

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)^3} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^6}{(2n+1)!} \times \frac{\left(5\,(n+1)^2 + 6\,(a-1)(n+1) + 2\,(a-1)^2\right)}{\prod_{k=0}^{n} (a+k)^4}.$$

Note: *Maple* establishes this identity as

$$-1/2\,\Psi(2,a) = -1/2\,\Psi(2,a) - \zeta(3) + 5/4\,{}_4F_3([1,1,1,1],[3/2,2,2],-1/4)$$

Hence

$$\zeta(4) = -\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\binom{2m}{m} m^4} + \frac{10}{3} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \sum_{k=1}^{m} \frac{1}{k}}{\binom{2m}{m} m^3}.$$

The case a=0 above is Apéry's formula for $\zeta(3)$!

Apéry-Like Relations Found Using Integer Relation Methods

$$\zeta(5) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2},$$

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}$$

$$\zeta(9) = \frac{9}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^9\binom{2k}{k}} - \frac{5}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2} + 5\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4} + \frac{45}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^6} - \frac{25}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\sum_{j=1}^{k-1}\frac{1}{j^2},$$

$$\zeta(11) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{11}\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4} - \frac{75}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^8} + \frac{125}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\sum_{i=1}^{k-1}\frac{1}{i^4}$$

Formulas for 7 and 11 were found by JMB and David Bradley; 5 and 9 by Kocher 25 years ago, as part of the general formula:

$$\sum_{k=1}^{\infty}\frac{1}{k(k^2-x^2)} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\frac{5k^2-x^2}{k^2-x^2}\prod_{m=1}^{k-1}\left(1-\frac{x^2}{m^2}\right)$$

Newer (2005) Results

Using **bootstrapping** and the "**Pade/pade**" function JMB and Dave Bradley then found the following remarkable result (1996):

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-x^4/k^4)} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}(1-x^4/k^4)} \prod_{m=1}^{k-1} \left(\frac{1+4x^4/m^4}{1-x^4/m^4}\right)$$

Following an analogous – but more deliberate – experimental-based procedure, we have obtained a similar general formula for ζ(2n+2) that is pleasingly parallel to above:

$$\sum_{k=1}^{\infty} \frac{1}{k^2-x^2} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}\left(1-x^2/k^2\right)} \prod_{m=1}^{k-1} \left(\frac{1-4\,x^2/m^2}{1-x^2/m^2}\right)$$

Note that this gives an *Apéry-like formula* for ζ(2n), since the LHS equals

$$\sum_{n=0}^{\infty} \zeta(2n+2)x^{2n} = \frac{1-\pi x \cot(\pi x)}{2x^2}$$

- We will sketch our experimental discovery of this in the new few slides.

The Experimental Scheme

1. We first supposed that $\zeta(2n+2)$ is a rational combination of terms of the form:

$$\sigma(2r;[2a_1,\cdots,2a_N]) \;:=\; \sum_{k=1}^{\infty}\frac{1}{k^{2r}\binom{2k}{k}}\prod_{i=1}^{N}\sum_{n_i=1}^{k-1}\frac{1}{n_i^{2a_i}}$$

where $r + a_1 + a_2 + \ldots + a_N = n + 1$ and a_i are listed increasingly.

2. We can then write:

$$\sum_{n=0}^{\infty}\zeta(2n+2)\,x^{2n} \stackrel{?}{=} \sum_{n=0}^{\infty}\sum_{r=1}^{n+1}\sum_{\pi\in\Pi(n+1-r)}\alpha(\pi)\,\sigma(\mathbf{2r};\mathbf{2}\pi)\,x^{2n}$$

where $\Pi(m)$ denotes the additive partitions of m.

3. We can then deduce that

$$\sum_{n=0}^{\infty}\zeta(2n+2)\,x^{2n} = \sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}(k^2-x^2)}P_k(x)$$

where $P_k(x)$ are polynomials **whose general form we hope to discover:**

The Bootstrap Process

$$\zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^2} = 3\sigma(2,[0]),$$

$$\zeta(4) = \mathbf{3}\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^4} - 9\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-2}}{\binom{2k}{k}k^2} = \mathbf{3}\sigma(4,[0]) - 9\sigma(2,[2])$$

$$\begin{aligned}\zeta(6) &= \mathbf{3}\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^6} - \mathbf{9}\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-2}}{\binom{2k}{k}k^4} - \frac{45}{2}\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-4}}{\binom{2k}{k}k^2}\\ &\quad + \frac{27}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\frac{\sum_{i=1}^{k-1} i^{-2}}{j^2\binom{2k}{k}k^2},\\ &= \mathbf{3}\sigma(6,[]) - \mathbf{9}\sigma(4,[2]) - \frac{45}{2}\sigma(2,[4]) + \frac{27}{2}\sigma(2,[2,2])\end{aligned}$$

$$\begin{aligned}\zeta(8) &= \mathbf{3}\sigma(8,[]) - \mathbf{9}\sigma(6,[2]) - \frac{45}{2}\sigma(4,[4]) + \frac{27}{2}\sigma(4,[2,2]) - 63\sigma(2,[6])\\ &\quad + \frac{135}{2}\sigma(2,[4,2]) - \frac{27}{2}\sigma(2,[2,2,2])\end{aligned}$$

$$\begin{aligned}\zeta(10) &= \mathbf{3}\sigma(10,[]) - \mathbf{9}\sigma(8,[2]) - \frac{45}{2}\sigma(6,[4]) + \frac{27}{2}\sigma(6,[2,2]) - \mathbf{63}\sigma(4,[6])\\ &\quad + \frac{135}{2}\sigma(4,[4,2]) - \frac{27}{2}\sigma(4,[2,2,2]) - \frac{765}{4}\sigma(2,[8]) + 189\sigma(2,[6,2])\\ &\quad + \frac{675}{8}\sigma(2,[4,4]) - \frac{405}{4}\sigma(2,[4,2,2]) + \frac{81}{8}\sigma(2,[2,2,2,2])\end{aligned}$$

Tabular Coefficients Obtained

Partition	Alpha	Partition	Alpha	Partition	Alpha
[empty]	3/1	1	-9/1	2	-45/2
1,1	27/2	3	-63/1	2,1	135/2
1,1,1	-27/2	4	-765/4	3,1	189/1
2,2	675/8	2,1,1	-405/4	1,1,1,1	81/8
5	-3069/5	4,1	2295/4	3,2	945/2
3,1,1	-567/2	2,2,1	-2025/8	2,1,1,1	405/4
1,1,1,1,1	-243/40	6	-4095/2	5,1	9207/5
4,2	11475/8	4,1,1	-6885/8	3,3	1323/2
3,2,1	-2835/2	3,1,1,1	567/2	2,2,2	-3375/16
2,2,1,1	6075/16	2,1,1,1,1	-1215/16	1,1,1,1,1,1	243/80
7	-49149/7	6,1	49140/8	5,2	36828/8

Partition	Alpha	Partition	Alpha	Partition	Alpha
5,1,1	-27621/10	4,3	32130/8	4,2,1	-34425/8
4,1,1,1	6885/8	3,3,1	-15876/8	3,2,2	-14175/8
3,2,1,1	17010/8	3,1,1,1,1	-1701/8	2,2,2,1	10125/16
2,2,1,1,1	-6075/16	2,1,1,1,1,1	729/16	1,1,1,1,1,1,1	-729/560
8	-1376235/56	7,1	1179576/56	6,2	859950/56
6,1,1	-515970/56	5,3	902286/70	5,2,1	-773388/56
5,1,1,1	193347/70	4,4	390150/64	4,3,1	-674730/56
4,2,2	-344250/64	4,2,1,1	413100/64	4,1,1,1,1	-41310/64
3,3,2	-277830/56	3,3,1,1	166698/56	3,2,2,1	297675/56
3,2,1,1,1	-119070/56	3,1,1,1,1,1	10206/80	2,2,2,2	50625/128
2,2,2,1,1	-60750/64	2,2,1,1,1,1	18225/64	2,1,1,1,1,1,1	-1458/64
1,1,1,1,1,1,1,1	2187/4480				

Resulting Polynomials (Ugly)

$$P_3(x) \approx 3 - \frac{45}{4}x^2 - \frac{45}{16}x^4 - \frac{45}{64}x^6 - \frac{45}{256}x^8 - \frac{45}{1024}x^{10} - \frac{45}{4096}x^{12} - \frac{45}{16384}x^{14} - \frac{45}{65536}x^{16}$$

$$P_4(x) \approx 3 - \frac{49}{4}x^2 + \frac{119}{144}x^4 + \frac{3311}{5184}x^4 + \frac{38759}{186624}x^6 + \frac{384671}{6718464}x^8 + \frac{3605399}{241864704}x^{10} + \frac{33022031}{8707129344}x^{12} + \frac{299492039}{313456656384}x^{14}$$

$$P_5(x) \approx 3 - \frac{205}{16}x^2 + \frac{7115}{2304}x^4 + \frac{207395}{331776}x^6 + \frac{4160315}{47775744}x^8 + \frac{74142995}{6879707136}x^{10} + \frac{1254489515}{990677827584}x^{12} + \frac{20685646595}{142657607172096}x^{14} + \frac{336494674715}{20542695432781824}x^{16}$$

$$P_6(x) \approx 3 - \frac{5269}{400}x^2 + \frac{6640139}{1440000}x^4 + \frac{1635326891}{5184000000}x^6 - \frac{5944880821}{18662400000000}x^8 - \frac{212874252291349}{67184640000000000}x^{10} - \frac{141436384956907381}{241864704000000000000}x^{12} - \frac{70524260274859115989}{870712934400000000000000}x^{14} - \frac{31533457168819214655541}{3134566563840000000000000000}x^{16}$$

$$P_7(x) \approx 3 - \frac{5369}{400}x^2 + \frac{8210839}{1440000}x^4 - \frac{199644809}{5184000000}x^6 - \frac{680040118121}{18662400000000}x^8 - \frac{278500311775049}{67184640000000000}x^{10} - \frac{84136715217872681}{241864704000000000000}x^{12} - \frac{22363377813883431689}{870712934400000000000000}x^{14} - \frac{5560090840263911428841}{3134566563840000000000000000}x^{16}$$

After Using "Padé" Function in Mathematica or Maple

$$P_1(x) \stackrel{?}{=} 3$$

$$P_2(x) \stackrel{?}{=} \frac{3(4x^2-1)}{(x^2-1)}$$

$$P_3(x) \stackrel{?}{=} \frac{12(4x^2-1)}{(x^2-4)}$$

$$P_4(x) \stackrel{?}{=} \frac{12(4x^2-1)(4x^2-9)}{(x^2-4)(x^2-9)}$$

$$P_5(x) \stackrel{?}{=} \frac{48(4x^2-1)(4x^2-9)}{(x^2-9)(x^2-16)}$$

$$P_6(x) \stackrel{?}{=} \frac{48(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-9)(x^2-16)(x^2-25)}$$

$$P_7(x) \stackrel{?}{=} \frac{192(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-16)(x^2-25)(x^2-36)}$$

which immediately suggests the general form:

$$\sum_{n=0}^{\infty} \zeta(2n+2)x^{2n} \stackrel{?}{=} 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2-x^2)} \prod_{m=1}^{k-1} \frac{4x^2-m^2}{x^2-m^2}$$

Several Confirmations of Z(2n+2)=Zeta(2n+2) Formula

- We symbolically computed the power series coefficients of the LHS and the RHS , and verified that they agree up to the term with $\mathbf{x^{100}}$.

- We verified that Z(1/6), Z(1/2), Z(1/3), Z(1/4) give numerically correct values (analytic values are known).

- We then affirmed that the formula gives numerically correct results for 100 pseudorandomly chosen arguments.

We subsequently proved this formula two different ways, including using the **Wilf-Zeilberger method**….

Apery summary

1. via PSLQ to 5,000 digits (120 terms)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \ldots$$

Euler (1707-73)

$$\mathcal{Z}(x) = 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2}$$

$$= \sum_{k=0}^{\infty} \zeta(2k+2) x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$$

$$= \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

2. reduced as hoped

2005 Bailey, Bradley & JMB *discovered and proved* - in 3Ms - three *equivalent* binomial identities

$$3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\binom{2k}{k}\left(k^2 - n^2\right)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

$$ {}_3F_2\left(\begin{matrix} 3n, n+1, -n \\ 2n+1, n+1/2 \end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily **computer proven** (Wilf-Zeilberger) **(now 2 human proofs)**

Apéry-Like Summations

There is a related formula whose lead term is $\zeta(4)$. So the following are all **seeds** for generating functions

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}},$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}},$$

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}.$$

Q3 Is there a corresponding generating function for $\zeta(4n+1)$?

Or more likely for the alternating zeta function

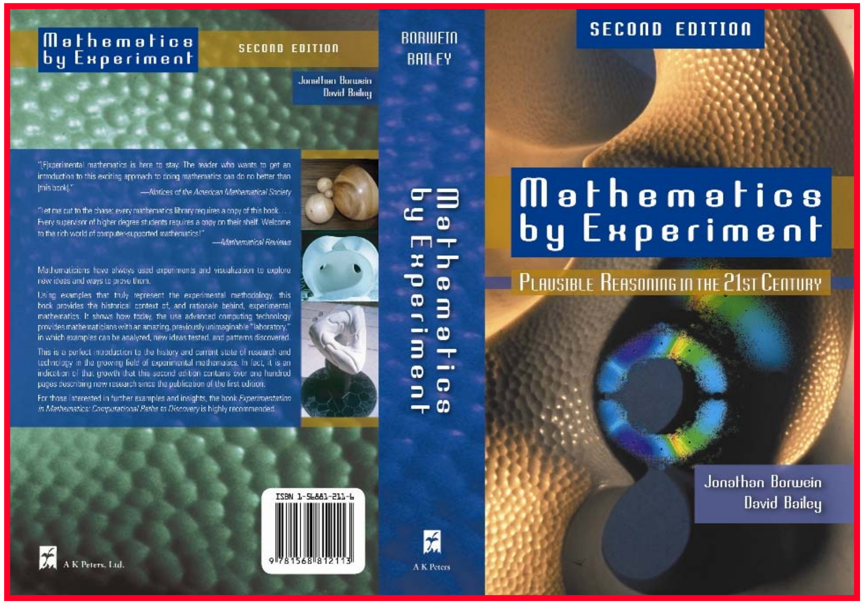
$$\alpha(s) := (1-2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

$$\alpha(1) = \log 2$$

Conclusions

New techniques now permit integrals, infinite series sums and other entities to be evaluated to high precision (hundreds or thousands of digits), thus permitting PSLQ-based schemes to discover new identities.

These methods typically do not suggest proofs, but often it is much easier to find a proof when one "knows" the answer is right.



Full details are in *Excursions in Experimental Mathematics*, and in some cases in the second edition of *Mathematics by Experiment* by Jonathan M. Borwein, and David H. Bailey. A "Reader's Digest" version of the later book(s) is available at **http://www.experimentalmath.info**

"The plural of 'anecdote' is not 'evidence'."
- Alan L. Leshner, *Science*'s publisher