Convex Analysis and Nonlinear Optimization, by

Jonathan M. Borwein and Adrian S. Lewis

This is an excellent modern book on convex and nonsmooth analysis, suitable for a text for a graduate class or for reference. In a relatively short space (the main part of the text is just 208 pages), the authors include a great deal of interesting material. Of course, their coverage is by necessity selective, as opposed to the more encyclopedic *Variational Analysis* by Rockafellar and Wets (Springer 1998), but by an extensive set of exercises a very nice picture, with many glimpses of intriguing related areas, is painted.

The unifying theme is optimality conditions, and these are treated in a succession of increasingly complicated settings, from smooth inequality-constrained optimization to convex, nonsmooth, and inequality- and equality-constrained problems. The variety of techniques used include max-functions and directional derivatives, value functions, and contingent cones. Indeed, the authors take care to point out the different approaches possible, and the interplay between algebraic/geometric and topological viewpoints. One of their goals is to introduce the reader to modern ideas in analysis through a study of optimization, and in this they succeed wonderfully. Besides developing general tools which are then used in optimization, they also apply optimality conditions to prove results that seemingly have nothing to do with minimizing or maximizing.

In a similar development, convex functions are first introduced in the real-valued case, and then more general extended real-valued convex functions appear, motivated by their relationship to optimization problems or analytic techniques already seen.

A recurring example is the case of spectral functions on symmetric matrices, which allows a very nice treatment of the currently exciting topic of semidefinite programming.

Besides the expected material in such a book, there is a section on convex process duality; a chapter on fixed-point theorems and variational analysis; and a chapter discussing infinite-dimensional extensions as well as characterizations of finite dimensionality via properties of convex sets and functions.

One remarkable feature is the requirement for substantial reader involvement: indeed the exercises are an integral part of the book. The preface claims that the main text is self-contained and the exercises optional, but a few times the exercises introduce definitions or results that are later used in the main text. Moreover, the exercises are very rich. Many sections are in fact shorter than the set of exercises they precede! Exercises contain proofs for (parts of) many of the results, and address such basic notions as recession cones, relative interiors, and infimal convolution, but also singular values, von Neumann's minimax theorem, Kirchoff's law, Schur- and order convexity, minimum volume ellipsoids, etc.

Because of the wealth of material in the exercises and the revisiting of many issues in more general contexts, easy navigation is important. Indeed, several of the more general optimality conditions appear in sections labelled "Value Functions" or "Generalized Derivatives." The authors have provided an extensive and very helpful index, and have supplemented it with a final chapter listing all named results (important theorems, in the main text and in the exercises, are all named for future reference) and notation.

My complaints are few: The authors explain that their concern is the analysis of optimization problems, not computational algorithms, which is fine; but their statement in the commentary on Section 7.1 that the use of penalty functions is now standard practice in computational optimization is somewhat misleading. Penalty functions are often used in a supplemental way, as merit functions, but penalty methods, where constrained problems are reduced to a sequence of unconstrained problems, are not: a distinction could have been made between these and interior-point methods which use barrier functions. The authors introduce the core of a set in Section 3.1, and stress that it is different from the interior. But at this point, almost all sets appearing are convex, and it is not until Section 4.1 that the authors explain (and prove) that the two notions coincide for convex sets. The book is very well prepared, and there seem to be very few typos, but one is noteworthy: the definition of the lim inf notation below the growth condition in Section 1.1 is inaccurate.

Despite these few complaints, this is a very rewarding book, and I highly recommend it to those who want to study the recent advances in convex and nonsmooth analysis in an engaging and stimulating text.