A review of "*Convex Analysis and Nonlinear Optimization. Theory and Examples*" by Jonathan M. Borwein and A. Lewis, CMS Books in Mathematics, Canadian Mathematical Society, Springer-Verlag, New York, 2000.

Convexity is one of the most ancient fields of study in mathematics: ancient Greeks devoted much attention to special convex polytopes, and in fact, the names these beautiful solids bear have been given by Greek mathematicians. At the same time, convexity plays a major role in modern mathematics, in particular through optimization problems. Why is it so?

A first explanation stems from the simplicity of the notion. Convex cones are the subsets of a vector space which retain most of the linear structure, next to vector subspaces: they are stable under addition and homotheties. When intersected with affine subspaces, they give rise to convex subsets (and there is a reverse construction, called homogenization).

A second reason lies in the fact that convex sets and convex functions enjoy some miraculous properties. Continuity of a convex function propagates all over the interior of its domain and it holds as soon as the function is locally bounded above. Directional derivatives exist at each point of the interior of the domain of the function. Closed convex sets (or functions) are also weakly closed, so that existence results in reflexive spaces are easily available under a coercivity condition. Finally, for convex functions, necessary optimality conditions are also sufficient.

A third insight into the reasons of the success of convexity can be obtained by the observation that one has access to a calculus apparatus within convexity which is rich and powerful. It even has some superiority over ordinary differential calculus, since rules for functions obtained as infima and suprema are available. Moreover, an alluring duality theory enables one to transform a function into a conjugate function and a convex minimization problem into a dual one which may help in solving the primal problem. Furthermore, the passage from an analytical viewpoint to a geometrical approach and the reverse passage are easy and fruitful. It consists in associating to a function its epigraph (i.e. the set of points above the graph of the function), which is a convex set whenever the function is convex; in the reverse direction, to any convex set C one can associate its indicator function ι_C which assigns 0 to any point of the set and the value $+\infty$ to any point outside the set. In a normed vector space, one can also use the distance function.

The features just described are also characteristic of the fields known as nonsmooth analysis and nonsmooth optimization. These topics are also treated in the book, in a way which makes it appear as a natural extension of convex analysis and convex optimization. Another opinion claims that convex analysis and convex optimization are parts of the field of nonsmooth analysis; it does not differ much of the preceding view: convex analysis is the root of nonsmooth analysis and its model. In both cases the substitute to the derivative of a function f at some point x is a set $\partial f(x)$ called the subdifferential. In both cases the analysis is one-sided and inequalities are more prevalent than equalities.

This fact indicates that both theories are able to treat problems of the real world in which inequalities and constraints are so much present.

In a remarkably concise form (about 220 pages), the authors expound the basic facts of convex analysis and constrained optimization. But they offer much more: with such a book, the reader gets the flavours of the most active themes of the field. This aim is reached through a wealth of exercises and commentaries which supplement the text. Part of this material has not appeared under a book form. Still, the exposition is gentle and intuitive enough and the grading of the exercises make them usable for teaching purposes. Although this book is not applications oriented, it contains a number of applicable topics (complexity, entropy, fixed points theory, spectral analysis of matrices, variational inequalities...) which make it suitable for courses in applied mathematics as well as in pure mathematics. In view of its emphasis on questions of optimization (optimality conditions of first and second order, duality, theorem of alternative...) it can be recommended for a course on optimization.

Among the most original subjects treated in the book are the ones dealt with in chapter 5: functions of eigenvalues of symmetric matrices, duality for semidefinite programming, convex process duality. This material can be considered as an appealing excursion out of the realm of classical linear algebra. The reader will be able to get a general view of the results considered in the book by looking at one of the two lists which end the book. Let us extract from this 13 pages-long list the following items: Fan's inequality, log-barrier functions, Rayleigh quotients, nearest points, BFGS updates, Ville's theorem, Schur-convexity, minimum volume ellipsoid, log-convexity, maximum entropy, Duffin duality gap, analytic center, Pareto minimization, tangency conditions, Fischer information function, condition number, KKM principle, cuscos... This short selection shows how lively are the topics illustrating the main results. It also shows that this list can prove to be useful in supplementing the index and the bibliography.

It may be surprising to those who know that the authors are leading researchers currently using the tools of functional analysis that they almost entirely limit their treatment to the finite dimensional case. They explain their choice in the preface by reasons of space and accessibility. In chapter 9, presented as a postscript, they indicate how finite dimensionality has played a critical role in the preceding chapters. They also point out the difficulties of extending the results displayed in the finite dimensional case by proposing a number of exercises and counterexamples. When infinite dimensional versions are possible, the adopted method of proof will enable the reader familiar with functional analysis to perform the extensions. Otherwise, the reader will find an infinite dimensional framework in the two recent references [A] and [Z] which may complete the present book and the classical monographs [R], [H], [L] and [H-U/L].

This book represents a tour de force for introducing so many topics of present interest in such a small space and with such clarity and elegance. It deserves to "make converts", a challenge the authors are about to win.

[A] D. Azé, *Analyse Convexe et Variationnelle*, Ellipses, Editions Marketing, Paris, 1997.

[H-U/L] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms*, Springer-Verlag, Berlin, 1993.

[H] R.B. Holmes, *Geometric Functional Analysis and its Applications*, Springer Verlag, New York, 1975.

[L] D.G. Luenberger, *Optimization by Vector Spaces Methods*, Addison Wesley, New York, 1969.

[R] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, N.J. 1970.

[Z] C. Zalinescu, *Convex Analysis in General Vector Spaces*, (expanded version of the Rumanian book, *Mathematical programming in infinite dimensional normed spaces*, Editura Academiei Romane, Bucharest, 1998) (submitted).